Julia set and Fatou set of entire solutions of complex differential equations

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Let $f(z) : \mathbb{C} \to \mathbb{C}$ be complex analytic. We define the iterates of $f$:

$$f^1 = f, \ldots, f^n = f \circ f^{n-1}, \ldots$$

The point $z$ is called normal if the family $\{f^n\}$ is normal in some neighbourhood of $z$, that is, every sequence in this family contains a locally uniformly convergent subsequence.

The Fatou set of $f$, denoted by $\mathcal{F}(f)$, is the set of normal points of the iterate family $\{f^n\}$.

Julia set $\mathcal{J}(f) = \mathbb{C} \setminus \mathcal{F}(f)$
Julia set and Fatou set

Examples: \( J(z^2) = \{ z : |z| = 1 \} \), \( J(e^z) = \mathbb{C} \)

(1) \( \mathcal{F}(f) \) is open, \( \mathcal{J}(f) \) is closed, non-empty and has no isolated points;

(2) \( \mathcal{F}(f), \mathcal{J}(f) \) are forward invariant and backward invariant;

(3) \( \mathcal{J}(f) = \mathbb{C} \), or \( \mathcal{J}(f) \) is nowhere dense (no interior points) in the plane. For polynomials of degree \( \geq 2 \), \( J(f) = \mathbb{C} \) never happens.
Limiting direction of Julia set

Qiao (1993, Chinese Acta. Math. Sinica.): Limiting direction of $\mathcal{J}(f)$ means a limit of the set \{arg $z_n | z_n \in \mathcal{J}(f)$ is an unbounded sequence\}

Zheng (2002, Bull. Austral. Math. Soc.): the Julia set has the radial distribution with respect to the radial arg $z = \theta$ if $\Omega(\theta - \varepsilon, \theta + \varepsilon) \cap \mathcal{J}(f)$ is unbounded for any $\varepsilon > 0$, where $\Omega(\theta - \varepsilon, \theta + \varepsilon) = \{z \in \mathbb{C} : \text{arg } z \in (\theta - \varepsilon, \theta + \varepsilon)\}$.

There exists an entire function whose Julia set has only one limiting direction arg $z = 0$. 
Limiting direction of Julia set

Baker (1965, J. London Math. Soc.): for any constant $A > 0$, there exists an entire function whose Julia set is contained in the region $\{|\text{Im } z| < A, \text{Re } z > 0\}$. The function is conjugate to

$$H(z) = \frac{1}{2\pi i} \int_L \frac{e^{e^t}}{t - z} dt + z - (1 + a)$$

where $L$ is the boundary of the region $\{\text{Re } z > 0, -\pi < \text{Im } z < \pi\}$ described in a clockwise direction.
Define the set
\[ \Delta(f) = \{ \theta \in [0, 2\pi) : \arg z = \theta \text{ is a limiting direction of } J(f) \}. \]
Clearly, \( \Delta(f) \) is closed, \( mes \Delta(f) \) is its linear measure.

Examples: 1. for polynomials \( p(z) \) with \( \deg(p) \geq 2 \), \( \Delta(p) = \emptyset \);
2. for \( 0 < \lambda \leq 1/e \), \( J(\lambda e^z) \subseteq \{ Rez > 1 \} \), so \( \Delta(\lambda e^z) = (0, \pi) \);
3. for \( \lambda > 1/e \), \( J(\lambda e^z) = \mathbb{C} \), so \( \Delta(\lambda e^z) = [0, 2\pi) \).
Lower order $\mu(f)$ and order $\sigma(f)$ of $f$ are defined by

$$
\mu(f) := \liminf_{r \to \infty} \frac{\log^+ T(r, f)}{\log r}, \quad \sigma(f) := \limsup_{r \to \infty} \frac{\log^+ T(r, f)}{\log r},
$$

where $T(r, f)$ is the Nevanlinna characteristic function of $f$.

Goldberg and Ostrovskii (Value Distribution on Meromorphic Functions, 2008): for any $\mu$ and $\lambda$ satisfying $0 \leq \mu \leq \lambda \leq \infty$, there is an entire function of order $\lambda$ and lower order $\mu$, which means that the lower order is different from the order.
Limiting direction of Julia set

The function $H(z)$ in Baker’s example is of infinite lower order.

Qiao (1994, Chinese Acta Math. Sinica.): If $f(z)$ is transcendental entire and $\mu(f) < \infty$, then

$$\text{mes} \Delta(f) \geq \min\{2\pi, \pi/\mu(f)\}.$$
Limiting direction of Julia set

The Julia sets of a transcendental entire function of finite lower order, its derivative and its primitive have a large amount of common radial distribution.

Qiao(2001, Ann. Acad. Sci. Fenn. Math.): Let $f$ be a transcendental entire function of lower order $\mu < \infty$. Then there exists a closed interval $I \subset \mathbb{R}$ such that all $\theta \in I$ are the common limiting directions of $J(f^{(n)})$, $n \in \mathbb{Z}$ and $mes I \geq \min\{2\pi, \pi/\mu\}$. Here $f^{(n)}$ denotes the $n$-th derivative or the $n$-th integral primitive of $f$ for $n \geq 0$ or $n < 0$ respectively.
Limiting direction of Julia set


Let $f(z)$ be a transcendental meromorphic function with
\[ \mu = \mu(f) < \infty \text{ and} \]
\[ \delta(\infty, f) = \liminf_{r \to \infty} \frac{m(r, f)}{T(r, f)} > 0, \]

where $m(r, f)$ is the proximity function of $f$. 
Limiting direction of Julia set

If $\mu = 0$, then $\Delta(f) = [0, 2\pi)$; if $\mu > 0$ and $J(f)$ has an unbounded component, then

$$\text{mes } \Delta(f) \geq \min \left\{ 2\pi, \frac{4}{\mu(f)} \arcsin \sqrt{\frac{\delta(\infty, f)}{2}} \right\}.$$ 

Qiu and Wu(2006, J. Austral. Math. Soc.): the conclusion still holds without the condition "$J(f)$ has an unbounded component".
Entire Functions of infinite lower order

1. \( f(z) = C_1 \cos(e^z) + C_2 \sin(e^z) \) with \( C_1, C_2 \) being arbitrary constants satisfies the equation \( f'' - f' + e^{2z} f = 0 \).

2. The Mathieu functions satisfy the Mathieu differential equation

\[
 f'' + (a - 2q \cos 2z) f = 0, \quad \cos z = \frac{1}{2}(e^{-iz} + e^{iz}),
\]

with given complex numbers \( a \) and \( q \).
Entire Functions of infinite lower order

3. Take $g_1(z) = \cosh z$ or $\sinh z$. Then

$$f(z) = e^{g_1} \left( C_1 + C_2 \int e^{-2g_1} dz \right)$$

satisfies the equation $f'' - (g_1 + (e^z - g_1)^2)f = 0$.

4. Airy functions $Ai(z)$ and $Bi(z)$ are two linearly independent solutions to $y'' - zy = 0$. Take $g_2(z) = Ai(z)$ or $Bi(z)$, then

$$f(z) = e^{g_2} \left( C_1 + C_2 \int e^{-2g_2} dz \right)$$

satisfies the equation $f'' - (zg_2 + (g_2')^2)f = 0$. 
5. \( J_n(z) \) and \( Y_n(z) \) denote Bessel functions of the first kind and the second kind respectively, and \( a = n + 1 \) and \( n \in \mathbb{N} \cup \{0\} \).

\[
f(z) = e^{-(a+1)z/2} \left( C_1 J_n(2e^{z/2}) + C_2 Y_n(2e^{z/2}) \right)
\]

satisfies the equation \( f'' + (n + 2)f' + e^z f = 0 \).
We first consider the linear differential equations

\[ f^{(n)} + A(z)f = 0, \]  

(1)

where \( A(z) \) is a transcendental entire function with finite order.


Let \( \{f_1, f_2, \ldots, f_n\} \) be the solution base of (1), and denote \( E = f_1 f_2 \cdots f_n \). Then

\[ \text{mes} \Delta(E) \geq \min\{2\pi, \pi/\sigma(A)\}. \]
Remark: Even though all non-zero solutions of (1) are entire and of infinite lower order, $E(z)$ may have finite lower order. For example, the equation $f'' - \frac{1}{4}(e^{2z} + 1)f = 0$ admits two linear independently solutions

$$f_1 = \exp\{-\frac{1}{2}(z + e^z)\}, \quad f_2 = \exp\{-\frac{1}{2}(z - e^z)\},$$

so $E(z) = e^{-z}$, and $\mu(E) = 1$. 
Solutions of complex differential equation

Next, we consider the general linear differential equations

\[ f^{(n)} + A_{n-1}f^{(n-1)} + \cdots + A_0f = 0. \]  \hspace{1cm} (2)


1. Let \( A_i(i = 0, 1, \cdots, n-1) \) be entire functions of finite lower order such that \( A_0 \) is transcendental and \( m(r, A_i) = o(m(r, A_0)) \) as \( r \to \infty \). Then \( \mu(f) = \infty \),

\[ mes \Delta(f) \geq \min\{2\pi, \pi/\mu(A_0)\}; \]
Solutions of complex differential equation

2. Let $A_i(i = 0, 1, \cdots, n - 1)$ be entire functions of finite order and $\sigma(A_i) < \sigma(A_0)(i = 1, 2, \cdots, n - 1)$. Then there exists a closed interval $I \in [0, 2\pi)$ such that $I \subset \Delta(f)$ and

$$mes I \geq \min\{2\pi, \pi/\sigma(A_0)\}.$$ 

Consider Qiao’s result in 2001, we investigate the common limiting directions of entire solutions, their derivatives and primitives.
Solutions of complex differential equation


1. Suppose that $A_i (i = 0, 1, \cdots, n - 1)$ are entire functions such that $A_0$ is transcendental and $T(r, A_j) = o(T(r, A_0))$ as $r \to \infty$. Then for every non-trivial solution $f$ of (2),

$$E(f) := \bigcap_{n \in \mathbb{Z}} \Delta(f^{(n)})$$

satisfies $mes E(f) \geq \min\{2\pi, \pi/\mu(A_0)\}$. 
Moreover, if $\sigma(A_j) < \sigma(A_0)$ ($j = 1, 2, \ldots, n - 1$), then there exists a closed interval $I \subseteq E(f)$ with $\text{mes } I \geq \min\{2\pi, \pi / \sigma(A_0)\}$.

2. Suppose that $A_0$ and $A_1$ are entire functions such that $A_0(z)$ is transcendental and $T(r, A_0) \sim \log M(r, A_0)$ as $r \to \infty$ outside a set of finite logarithmic measure, $A_1(z)$ has a finite deficient value $a$, i.e. $\delta(a, A_1) > 0$. For every non-trivial solution $f$ of

$$f'' + A_1(z)f' + A_0(z)f = 0,$$  \hfill (3)
Solutions of complex differential equation

we have

\[ \text{mes } E(f) \geq \min \left\{ 2\pi, \frac{4}{\mu(A_1)} \arcsin \sqrt{\frac{\delta(a, A_1)}{2}} \right\}, \]

Moreover, if \( \sigma(A_1) < \infty \), then \( \mu(f) = \infty \).

Remark: There exist entire functions satisfying the hypothesis on \( A_0 \), for example, entire functions having Fejér gaps. Here, the function \( g(z) = \sum_{n=1}^{\infty} a_n z^{\lambda_n} \) is said to having Fejér gaps if \( \sum_{n=1}^{\infty} \lambda_n^{-1} < \infty \).
Solutions of complex differential equation

Finally, if there is no dominant coefficient, we consider the differential equations with exponential coefficients

\[ f'' + (A_1(z)e^{P_1(z)} + B_1(z))f' + (A_2(z)e^{P_2(z)} + B_2(z))f = 0, \quad (4) \]

where \( A_j(\neq 0), B_j \) are entire, and \( P_j = a_j z^{k_j} + \cdots, (j = 1, 2) \) are two polynomials of degree \( k_j \geq 0 \).

Here, \( \max\{\sigma(A_j), \sigma(B_j)\} < k_j (j = 1, 2) \) for \( k_j > 0 \), and \( A_j \) and \( B_j \) are constants for \( k_j = 0 \).

Suppose that \( k_1 + k_2 \neq 0 \). For every non-zero solution \( f \) of (4), we have \( \mu(f) = \infty \) in the following three cases, and

1. if \( k_1 < k_2 \), then \( \text{mes}E(f) \geq \pi \); 
2. if \( k_1 = k_2 \) and \( a_1/a_2 = b < 1 \), then \( \text{mes}E(f) \geq \pi \); 
3. if \( k_1 = k_2 \) and \( a_1/a_2 = b \notin \mathbb{R} \), then

\[
\text{mes}E(f) \geq \min \{ \arg b, 2\pi - \arg b \}.
\]
Remark: 1. If $k_1 < k_2$, $A_2 e^{P_2} + B_2$ is the dominant coefficient. It follows from Huang and Wang (2014) that $mes\Delta(f) \geq \pi/k_2$. By the conclusion 1, we obtain $mesE(f) \geq \pi$, which is better when $k_2 \geq 2$.

2. Note that $f''' + e^z f' - e^z f = 0$ admits the particular solution $f_0 = e^z + 1$, it implies that the condition $a_1/a_2 < 1$ in the conclusion 2 can not be omitted.
Wang and Chen (2016): Let $f(\neq 0)$ be a solution of the Mathieu differential equation

$$f'' + (a - 2q \cos 2z)f = 0,$$

where $a, q$ are given constants. If $q \neq 0$, then $\mu(f) = \infty$ and $\text{mes}E(f) = 2\pi$.

Remark: Clearly, $\mu(a - 2q \cos 2z) = 1$ for $q \neq 0$. For this case, the result is better than $\text{mes} \Delta(f) \geq \pi$ by Huang and Wang (2014).
Baker wandering domain

For a connected component $U$ of $\mathcal{F}(f)$, $f^n(U)$ is contained in a component of $\mathcal{F}(f)$, denoted by $U_n$. If $U_n \neq U_m$ for $n \neq m$ ($U$ is never periodic or eventually periodic), then $U$ is called the wandering domain of $f$.

Sullivan: A rational function whose degree is at least 2 has no wandering domain.

And $U$ is called the Baker wandering domain if $U$ is wandering, and all $U_n$ are multiply-connected and surround 0, and the Euclidean distance $dist(0, U_n) \to +\infty$ as $n \to +\infty$. 
Example for Baker wandering domain given by Baker (1976, J. Austral. Math. Soc.): Let $g(z) = Cz^2 \prod_{n=1}^{\infty} (1 + \frac{z}{a_n})$ where the $a_n$ satisfy $1 < a_1 < a_2 < \cdots$ and $a_{n+1} < g(a_n) < 2a_{n+1}$. The $a_n$ are also chosen so that annuli $A_n: a_n^2 < |z| < a_{n+1}^2$ have the property $g(A_n) < A_{n+1}$. The component $U_n$ of $\mathcal{F}(g)$ which contains $A_n$ is the Baker wandering domain.
Baker wandering domain

Baker(1984, Proc. London Math. Soc.): For a transcendental entire function $f(z)$, every multiply-connected component of $F(f)$ must be Baker wandering. In this case, $F(f)$ and $J(f)$ both have only bounded components.

If $f(z)$ has a finite asymptotic value (Baker, 1984), or if $f(z)$ has a finite Nevanlinna deficient value (Zheng, Math. Proc. Camb. Phil. Soc., 2002), then $f(z)$ has only simply connected Fatou components.
Let \( f(z) \) be a transcendental meromorphic function with at most finitely many poles. Three interesting criteria from Zheng(2006, J. Math. Anal. Appl.):

1. If \( \mathcal{J}(f) \) has only bounded components, then for any complex number \( a \), there exists a constant \( 0 < d < 1 \) and two sequences \( \{r_n\} \) and \( \{R_n\} \) with \( r_n \to \infty \) and \( R_n/r_n \to \infty \) \((n \to \infty)\) such that

\[
M(r, a, f)^d \leq L(r, d, f), \quad r \in G = \bigcup_{n=1}^{\infty} (r_n, R_n)
\]

where \( M(r, a, f) = \max\{|f(z)| : |z - a| = r\} \), \( L(r, a, f) = \min\{|f(z)| : |z - a| = r\} \), \( G \) has an infinite logarithmic measure.
Baker wandering domain

2. If for all sufficiently large $r > 0$ and $d > 1$, we have

$$\log M(2r, f) > d \log M(r, f),$$

then $\mathcal{J}(f)$ has an unbounded component and $f(z)$ has no Baker wandering domain.

3. Every transcendental meromorphic functions satisfying linear differential equation with rational coefficients must have no the Baker wandering domain.
Solutions of complex differential equations

We also consider the equation $f'' + A(z)f = 0$, where $A(z)$ is entire and of finite order.


Suppose that a real function $\delta(\theta)$ is continuous except finitely many points, it has $m(\geq 1)$ zeros outside the discontinuity points. There exists a set $H$ of finitely many element and a set $G$ of finite logarithmic measure. For any $\theta \in [0, 2\pi) \setminus H$, there is $R > 0$ such that $|z| = r > R$ and $r \notin G$, we have either of
Solutions of complex differential equations

(A1) if $\delta(\theta) > 0$, then $|A(re^{i\theta})| \geq \exp\{c_1\delta(\theta)r^{d_1}\}$;

(A2) if $\delta(\theta) < 0$, then $|A(re^{i\theta})| \leq \exp\{c_2\delta(\theta)r^{d_2}\}$,

where $c_i, d_i (i = 1, 2)$ are positive constants. Then $\mu(E) = \infty$, and $E(z)$ have no Baker wandering domains, that is, $E(z)$ has only simply-connected Fatou components.
Remark: we could take $A(z)$ as the following form

(1) $A(z) = B(z)e^{p(z)}$ where $p(z)$ is a non-constant polynomial, and $B(z)$ is an entire function satisfying $\rho(B) < \deg(p)$;

(2) $A(z) = P(e^{p(z)})$ where $p(z)$ is a non-constant polynomial, and $P(z)$ is a polynomial without constant term;

(3) $A(z) = B_1(z)e^{p_1(z)} + B_2(z)e^{p_2(z)}$ where $p_i(z) = a_i z^n + \cdots$ are two non-constant polynomials with $\deg(p_1) \neq \deg(p_2)$ or $a_1 \neq a_2$ for $\deg(p_1) = \deg(p_2)$, $B_i(z)(i = 1, 2)$ are entire functions of order less than $\deg(p_i)$.
Recently, we also consider the equation $f'' + Af = F$ where entire function $A$ satisfies $(A1)$ and $(A2)$, $F$ is a polynomial. Then we prove that for every nontrivial solution $f$, $\mu(f) = \infty$ and all $f^{(n)}(n \in \mathbb{Z})$ have no Baker wandering domain.
Solutions of complex differential equations

Wang and Chen (2016): Suppose that $B_j (j = 1, 2)$ are constants, and either of the following two conditions holds:

(1) $k_1 < k_2$,

(2) $k_1 = k_2$ and $a_1/a_2 = b \notin \mathbb{R}$ or $b \in (0, 1)$.

Then for every solution $f (\not\equiv 0)$ of (4), all $f^{(n)} (n = 0, \pm 1, \pm 2, \cdots)$ have no Baker wandering domain, that is, they only have simply connected Fatou component.
tools in the proof

Nevanlinna theory in an angle \( \Omega(\alpha, \beta) = \{ z : \arg z \in (\alpha, \beta) \} \):

Let \( g(z) \) be entire on the closure of \( \Omega(\alpha, \beta) \) where \( \beta - \alpha \in (0, 2\pi] \), \( \omega = \pi / (\beta - \alpha) \), we define

\[
A_{\alpha,\beta}(r, g) = \frac{\omega}{\pi} \int_{1}^{r} \left( \frac{1}{t^\omega} - \frac{t^\omega}{r^{2\omega}} \right) \left\{ \log^+ |g(te^{i\alpha})| + \log^+ |g(te^{i\beta})| \right\} \frac{dt}{t},
\]

\[
B_{\alpha,\beta}(r, g) = \frac{2\omega}{\pi r^\omega} \int_{\alpha}^{\beta} \log^+ |g(re^{i\theta})| \sin \omega(\theta - \alpha) d\theta.
\]
The Nevanlinna’s angular characteristic of \( g \) is defined by

\[
S_{\alpha, \beta}(r, g) = A_{\alpha, \beta}(r, g) + B_{\alpha, \beta}(r, g),
\]

and we use \( \sigma_{\alpha, \beta}(g) \) to denote the order of \( S_{\alpha, \beta}(r, g) \), that is

\[
\sigma_{\alpha, \beta}(g) = \limsup_{r \to \infty} \frac{\log S_{\alpha, \beta}(r, g)}{\log r}.
\]

Set \( \Omega(r; \alpha, \beta) = \Omega(\alpha, \beta) \cap \{z \in \mathbb{C} : |z| > r\} \).
Lemma 1 (Zheng, Wang and Huang 2002): Let $f(z) : \Omega(r_0, \theta_1, \theta_2) \to U$ be analytic, $U$ a hyperbolic domain ($\mathbb{C} \setminus U$ has at least three points). If there exists a point $a \in \partial U \setminus \{\infty\}$, such that $C_U(a) = \inf \{\lambda_U(z)|z-a| : \forall z \in U\} > 0$, where $\lambda_U(z)$ is the hyperbolic density on $U$, then there exists a constant $d > 0$ such that for sufficiently small $\varepsilon > 0$, we have

$$|f(z)| = O(|z|^d), \ z \to \infty, \ z \in \Omega(r_0, \theta_1 + \varepsilon, \theta_2 - \varepsilon).$$

**Remark.** It is well known that if every component of hyperbolic domain $U$ is simply connected, then $C_U(a) \geq 1/2$. 
Lemma 2 (Zheng’s book, Value distribution of meromorphic functions) Let $f(z)$ be a meromorphic function on $\Omega(\alpha - \varepsilon, \beta + \varepsilon)$ for $\varepsilon > 0$ and $0 < \alpha < \beta < 2\pi$. Then

$$A_{\alpha,\beta}(r, \frac{f'}{f}) + B_{\alpha,\beta}(r, \frac{f'}{f}) \leq K \left( \log^+ S_{\alpha-\varepsilon,\beta+\varepsilon}(r, f) + \log r + 1 \right),$$

for $r > 1$ possibly except a set with finite linear measure.

$\bigcup_{n=1}^{\infty} B(z_n, r_n)$ is called an $R$-set if $\sum_{n=1}^{\infty} r_n < \infty$ and $z_n \to \infty$ as $n \to \infty$.

The case $n = 1$ and $n = 2$ of the following lemma could be found in Mokhon’ko (1989, Ukrain. Mat. Zh.) and S. J. Wu (1994, Math. Scand.) respectively.
Lemma 3 (Huang and Wang 2012, J. Math. Anal. Appl.) Suppose that \( g(z) \) is analytic in \( \Omega(r_0, \alpha, \beta) \) with \( \sigma_{\alpha,\beta}(g) < \infty \). Choose \( \alpha < \alpha_1 < \beta_1 < \beta \). Then for every \( \varepsilon_j \in (0, \frac{\beta_j - \alpha_j}{2}) \) \((j = 1, 2, \cdots, n - 1)\) outside a set of linear measure zero with

\[
\alpha_j = \alpha + \sum_{s=1}^{j-1} \varepsilon_s, \quad \beta_j = \beta - \sum_{s=1}^{j-1} \varepsilon_s, \quad j = 2, 3, \cdots, n - 1,
\]

there exist \( K > 0 \) and \( M > 0 \) only depending on \( g, \varepsilon_1, \cdots, \varepsilon_{n-1} \) such that

\[
\left| \frac{g^{(n)}(z)}{g(z)} \right| \leq Kr^M \left( \sin k(\psi - \alpha) \prod_{j=1}^{n-1} \sin k_j(\psi - \alpha_j) \right)^{-2},
\]

for all \( z \in \Omega(\alpha_{n-1}, \beta_{n-1}) \) outside a \( R \)-set \( D \), where \( k = \pi/(\beta - \alpha) \) and \( k_j = \pi/(\beta_j - \alpha_j) \) \((j = 1, 2, \cdots, n - 1)\).
Lemmas in the proof

Lemma 4 (Baernstein 1973, Proc. London Math. Soc.) Let $f(z)$ be a transcendental meromorphic function of finite lower order $\mu$, and have one deficient value $a$. Let $\Lambda(r)$ be a positive function with $\Lambda(r) = o(T(r, f))$ as $r \to \infty$. Then for any fixed sequence of Pólya peaks $\{r_n\}$ of order $\mu$, we have

$$
\liminf_{r \to \infty} \text{mes} D_{\Lambda}(r_n, a) \geq \min \left\{2\pi, \frac{4}{\mu} \arcsin \sqrt{\frac{\delta(a, f)}{2}} \right\},
$$

where $D_{\Lambda}(r, \infty) = \{\theta \in [-\pi, \pi) : |f(re^{i\theta})| > e^{\Lambda(r)}\}$, and

$$
D_{\Lambda}(r, a) = \{\theta \in [-\pi, \pi) : |f(re^{i\theta}) - a| < e^{-\Lambda(r)}\}.
$$
Lemma 5 (Bellman, 1953, Stability theory on differential equations)

Let $p_j(x)(j = 1, 2, \cdots, n)$ and $f(x)$ be a continuous complex-valued functions on the interval $[a, b]$, and let $P_j(x)(j = 1, 2, \cdots, n)$ and $F(x)$ be non-negative continuous functions with $|p_j(x)| \leq P_j(x)$ and $|f(x)| \leq F(x)$. Suppose that $v(x)$ and $V(x)$ satisfy the following equations

$$v^{(n)} - \sum_{j=1}^{n} p_j(x)v^{(n-j)} = f(x),$$
$$V^{(n)} - \sum_{j=1}^{n} P_j(x)V^{(n-j)} = F(x),$$

respectively. Then if $V^{(k)}(a) \geq |v^{(k)}(a)| (k = 0, 1, \cdots, n - 1)$, we have

$$|v^{(k)}(x)| \leq V^{(k)}(x), \quad \text{for } x \in [a, b].$$
1. Limiting direction of Julia set: We would prove by reduction to absurdity.

Firstly, by Lemma 1, we find some angles $\Omega(r_0, \alpha_i, \beta_i)$ such that $|f^{n_j}| = O(|z|^d)$. Then applying Lemma 2 and Lemma 3, we obtain

$$\left| \frac{f^{(s)}(z)}{f(z)} \right| \leq K r^M, \quad (s = 1, 2, \ldots, n - 1)$$

for the smaller angles $\Omega(r_0, \alpha_i + \varepsilon, \beta_i - \varepsilon)$ outside a R-set.

Now using the spread relation on the Pólya peak, i.e. Lemma 4, on $A_0$. Note that $A_0$ is entire, so $\delta(\infty, A_0) = 1$. Take
sketch of the proof

\[ \Lambda(r) := \max \left\{ \sqrt{\log r}, \sqrt{T(r, A_i)}, i = 1, 2, \cdots, n - 1 \right\} \sqrt{T(r, A_0)}. \]

We could find an interval \( I \subset (\alpha_i, \beta_i) \) such that \( F_k = I \cap D_{\Lambda(r_k)} \) with \( \text{mes}(F_k) \geq c > 0 \) for infinitely many \( k \). Thus

\[
\int_{F_k} \log^+ |A_0(r_k e^{i\theta})|d\theta \geq c\Lambda(r_k).
\]

Rewrite equation (2) as

\[ A_0 = -\left( \frac{f^{(n)}}{f} + A_{n-1} \frac{f^{(n-1)}}{f} + \cdots + \frac{f'}{f} A_1 \right). \]
sketch of the proof

Then by integration, we have

\[ c \Lambda(r) \leq \int_{F_k} \log^+ |A_0(r_k e^{i\theta})| d\theta \leq \sum_{i=1}^{n-1} m(r_k, A_i) + O(\log r_k). \]

which is impossible since \(A_0\) is transcendental and
\(T(r, A_i) = o(T(r, A_0))(i = 1, 2, \cdots, n - 1)\) as \(r \to \infty\).
2. Baker wandering domain: We could find some angle \( \Omega(\alpha, \beta) \) such that \( |A(z)| \leq M \) on such angle. Then by Lemma 5 and Lemma 2, we conclude that \( S_{\alpha+\varepsilon, \beta-\varepsilon}(r, f^{(n)}) = r \).

We assume that \( g = f^{(n)} \) has no Baker wandering domain, it follows from Zheng’s criteria that, there exists 0 < \( d < 1 \), \( |g(z)| \geq M(r, g)^d \) where \( r \in G_0 \), \( G_0 \) is a set of infinite logarithmic measure. By the definition of \( B_{\alpha, \beta}(r, g) \), we conclude that \( \log M(r, g) \leq O(r^\omega) \), which contradicts with \( \mu(f) = \infty \).
Thanks for your attention!