An Algebraic Geometric Approach to Multidimensional Symbolic Dynamics

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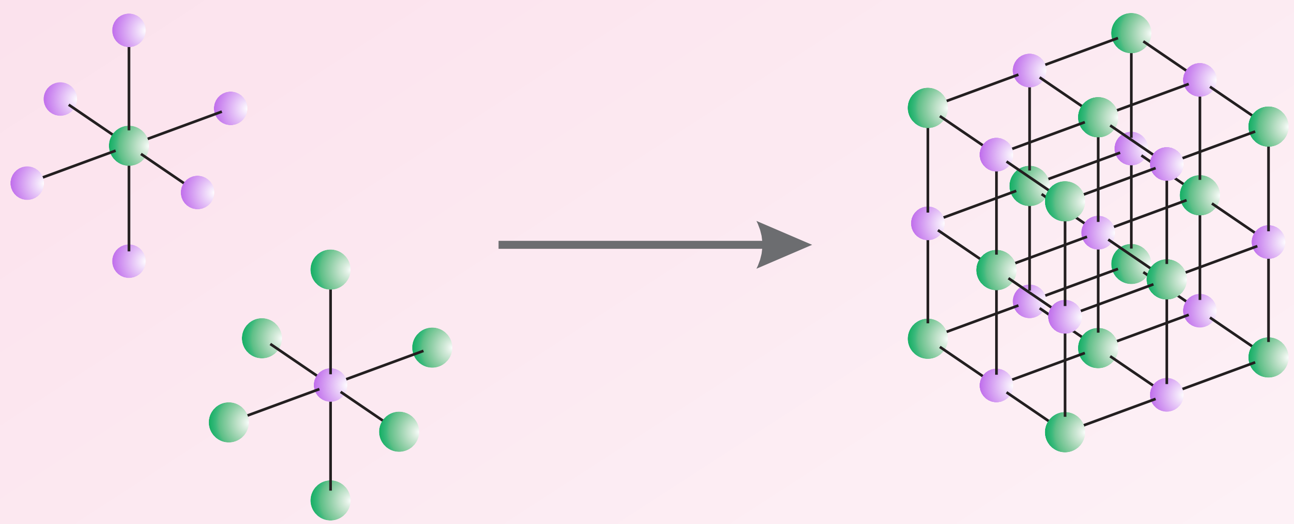
We study how **local constraints** enforce **global regularities**

This is a common phenomenon is sciences. For example, formation of crystals:

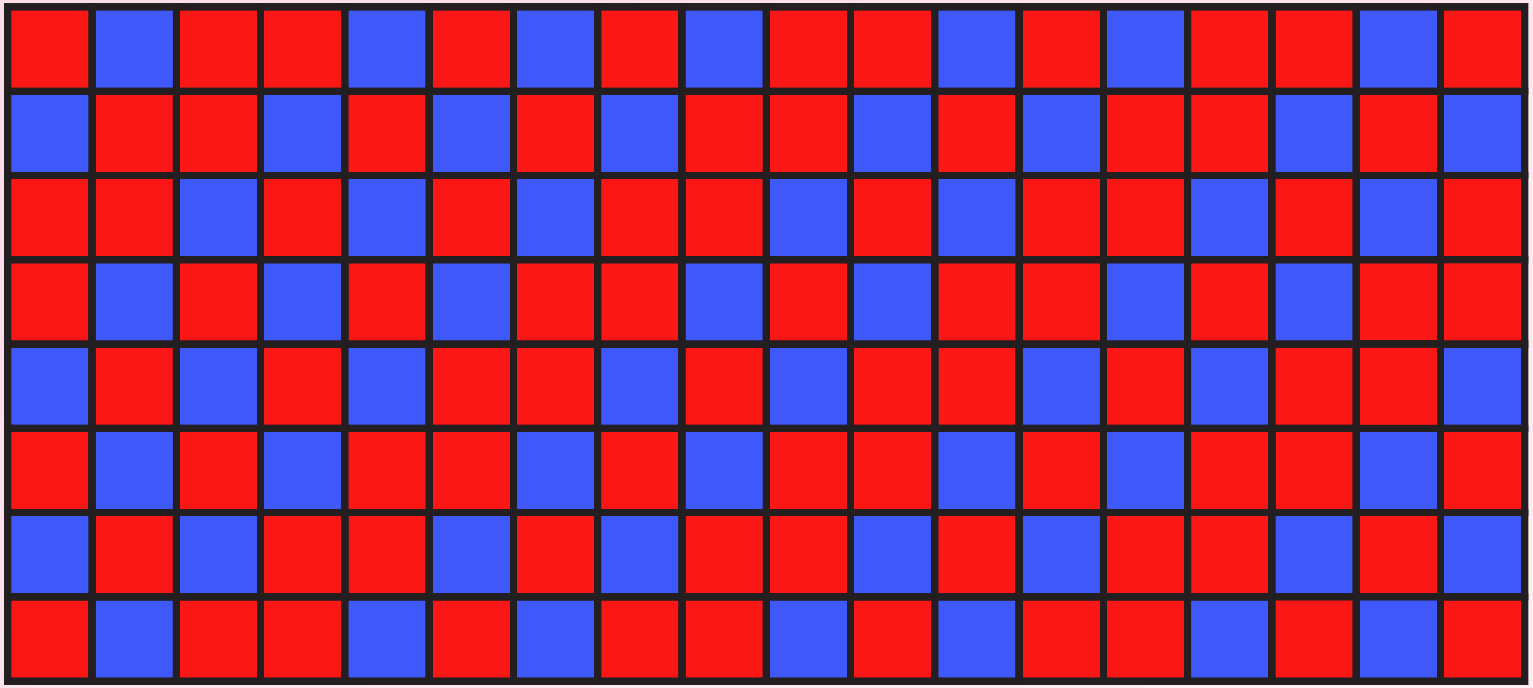


Atoms attach to each other in a limited number of ways

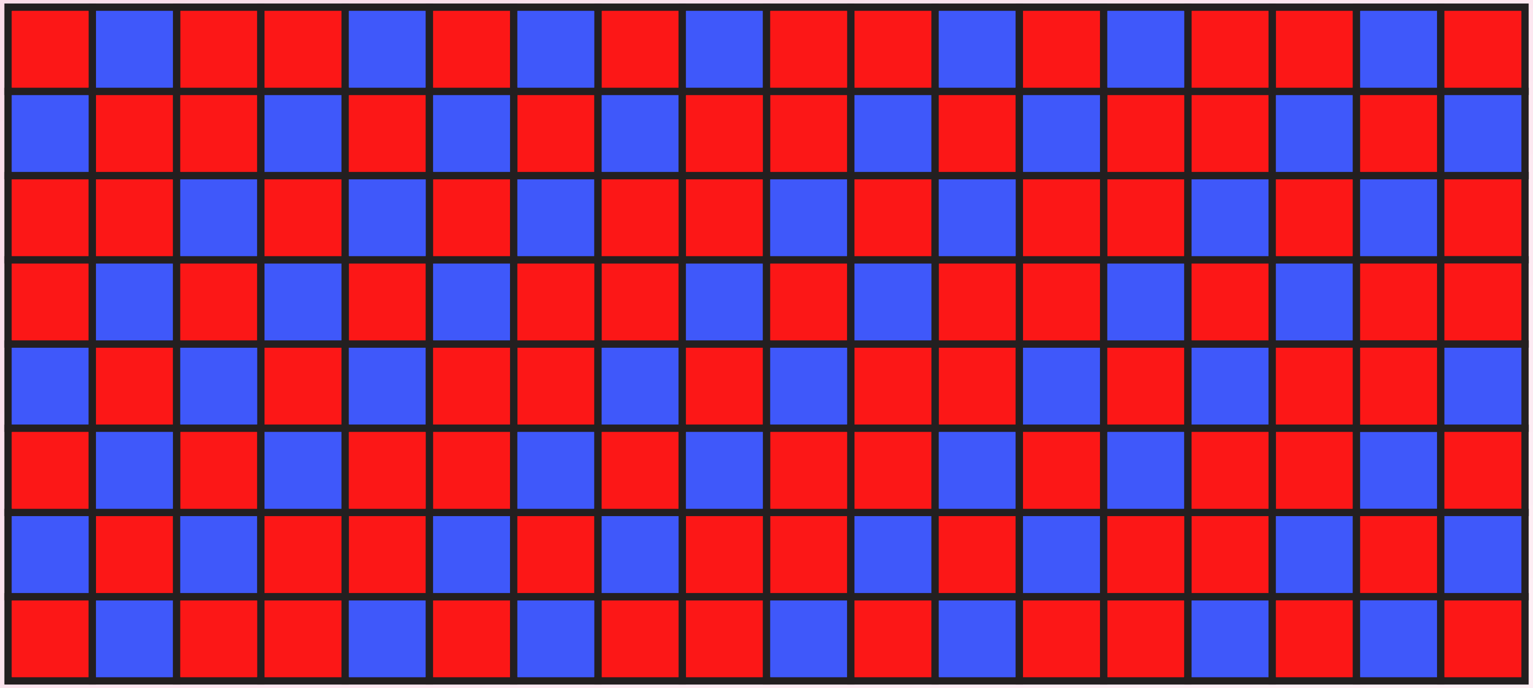
$\Longrightarrow$ periodic arrangement of the atoms

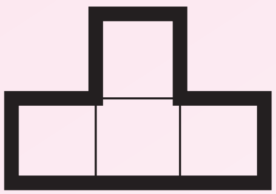
Our goal is to understand **fundamental underlying principles** that connect local rules to the global regularities observed in the structures.

Our setup: multidimensional symbolic dynamics (=tilings)

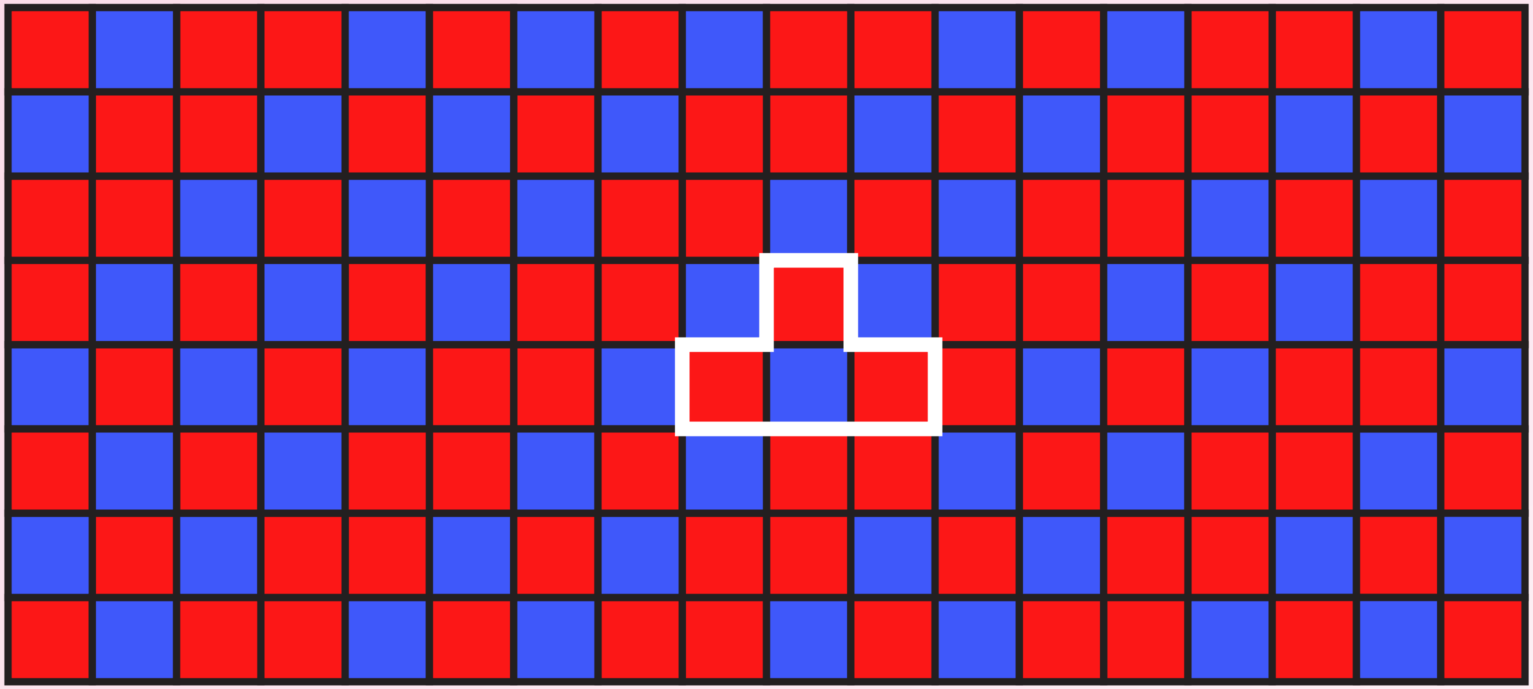


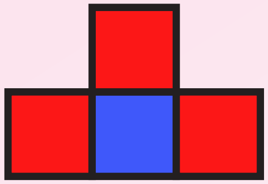
Configurations are infinite d-dimensional grids of symbols.





For a fixed finite shape D, we observe the **D-patterns** in the configuration.





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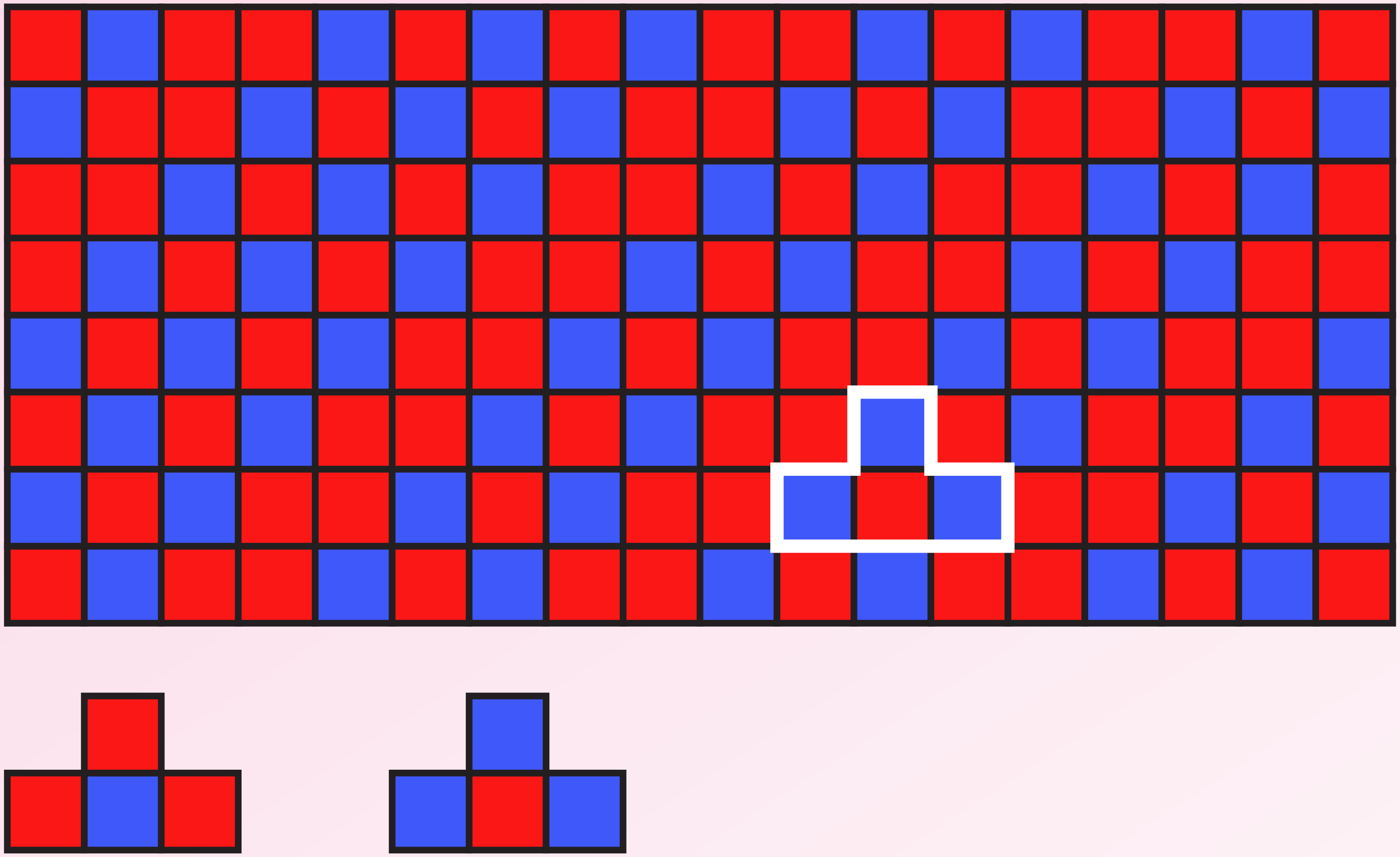

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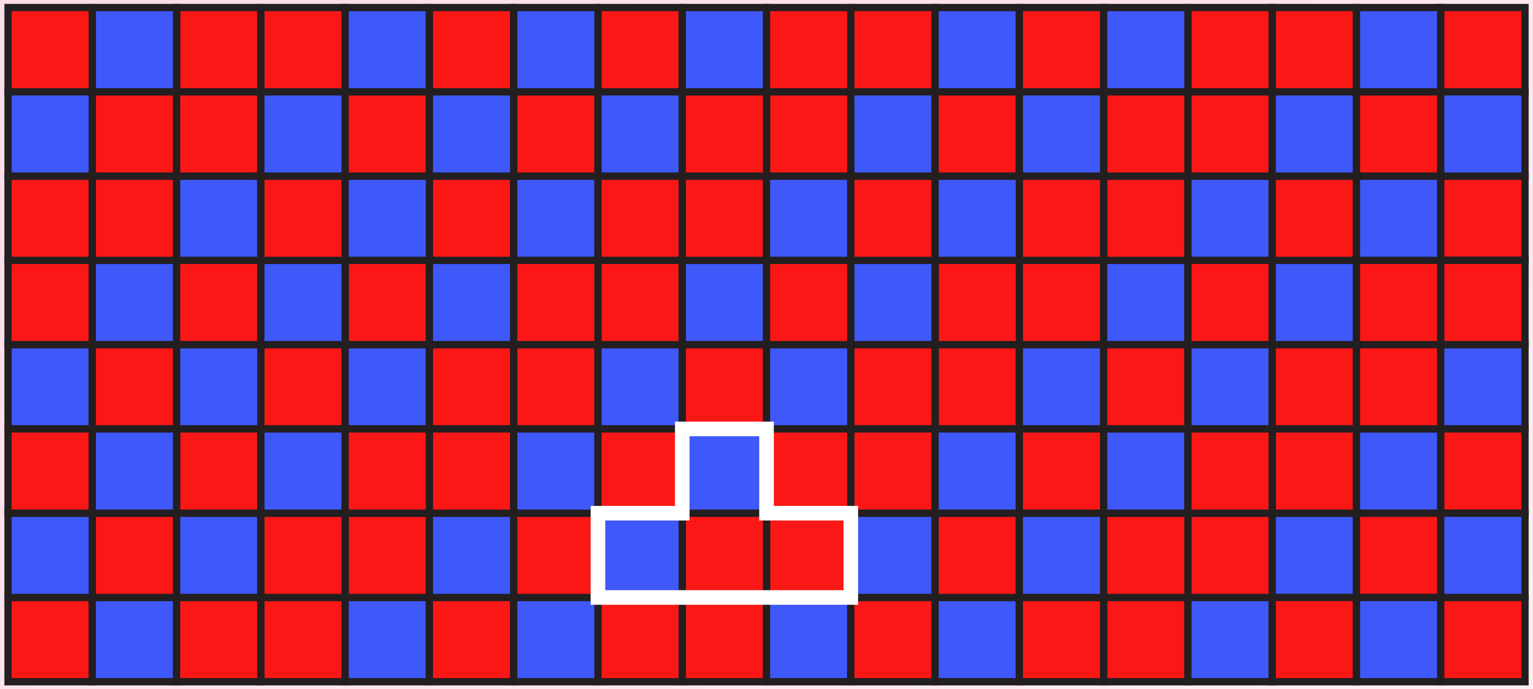

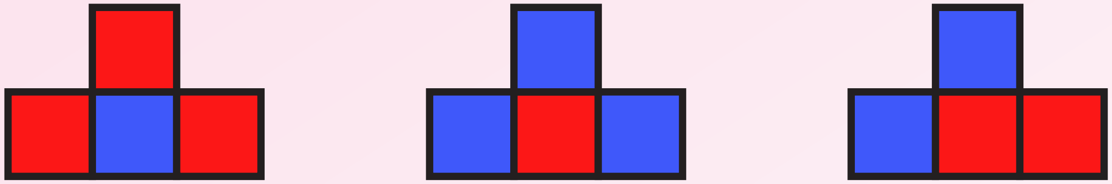

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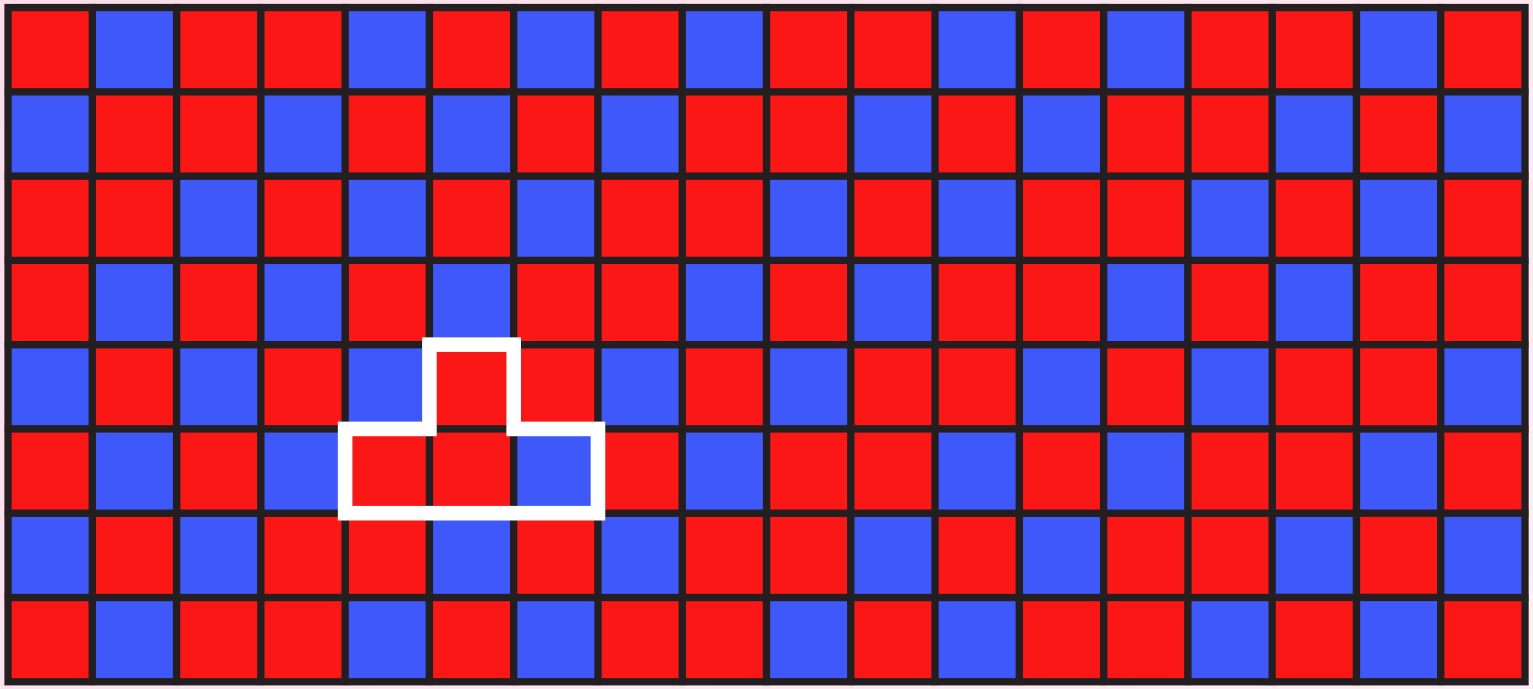

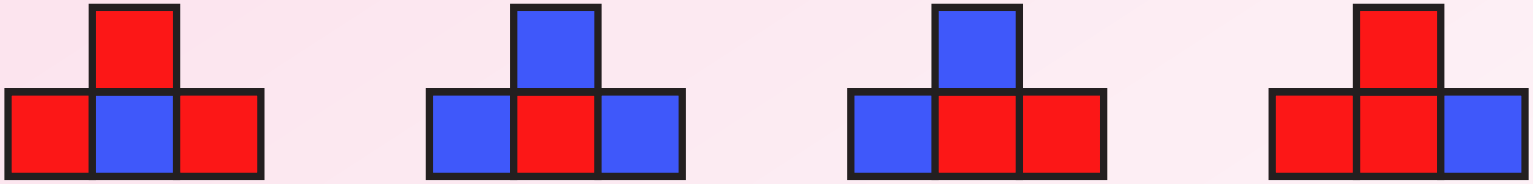

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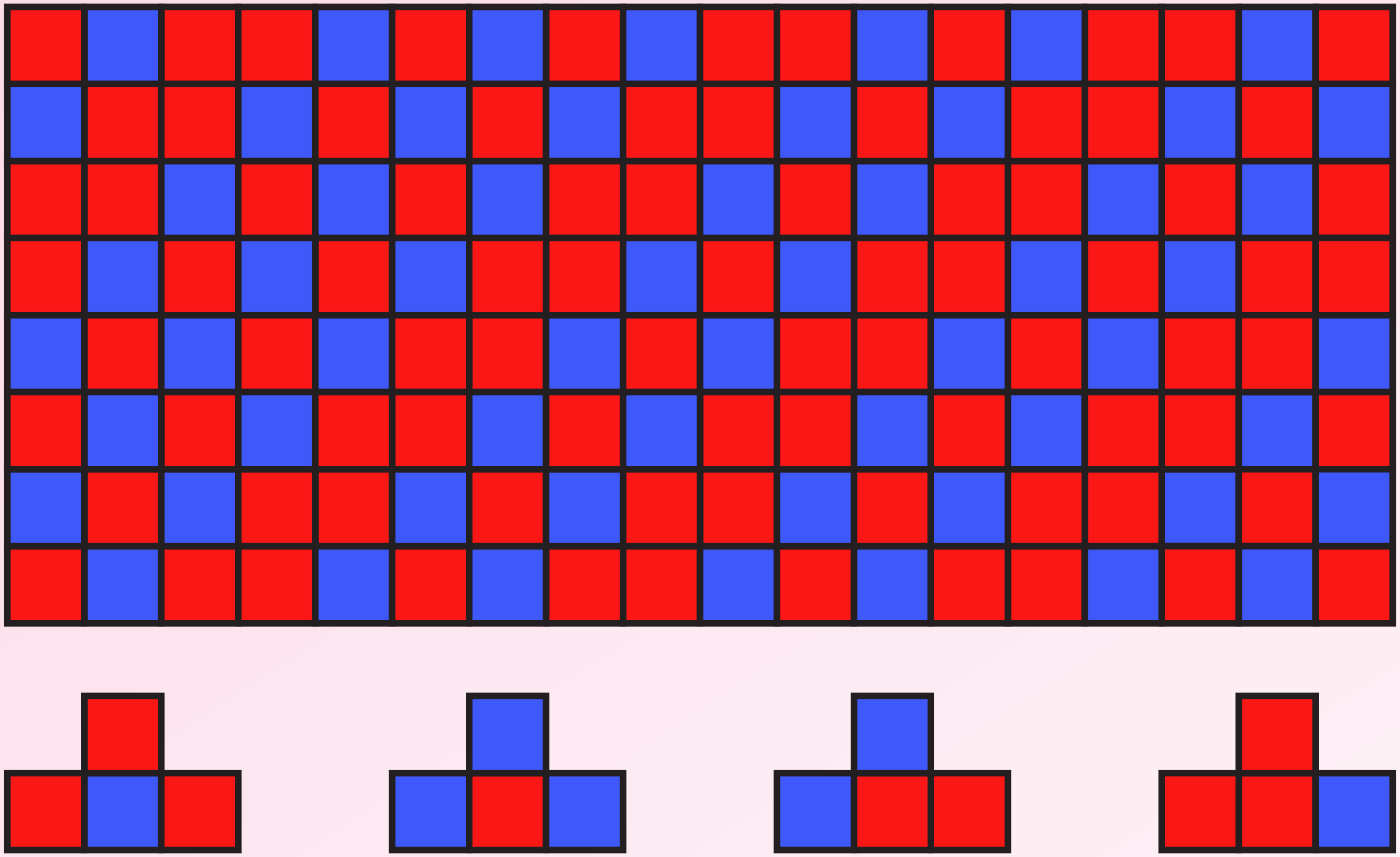
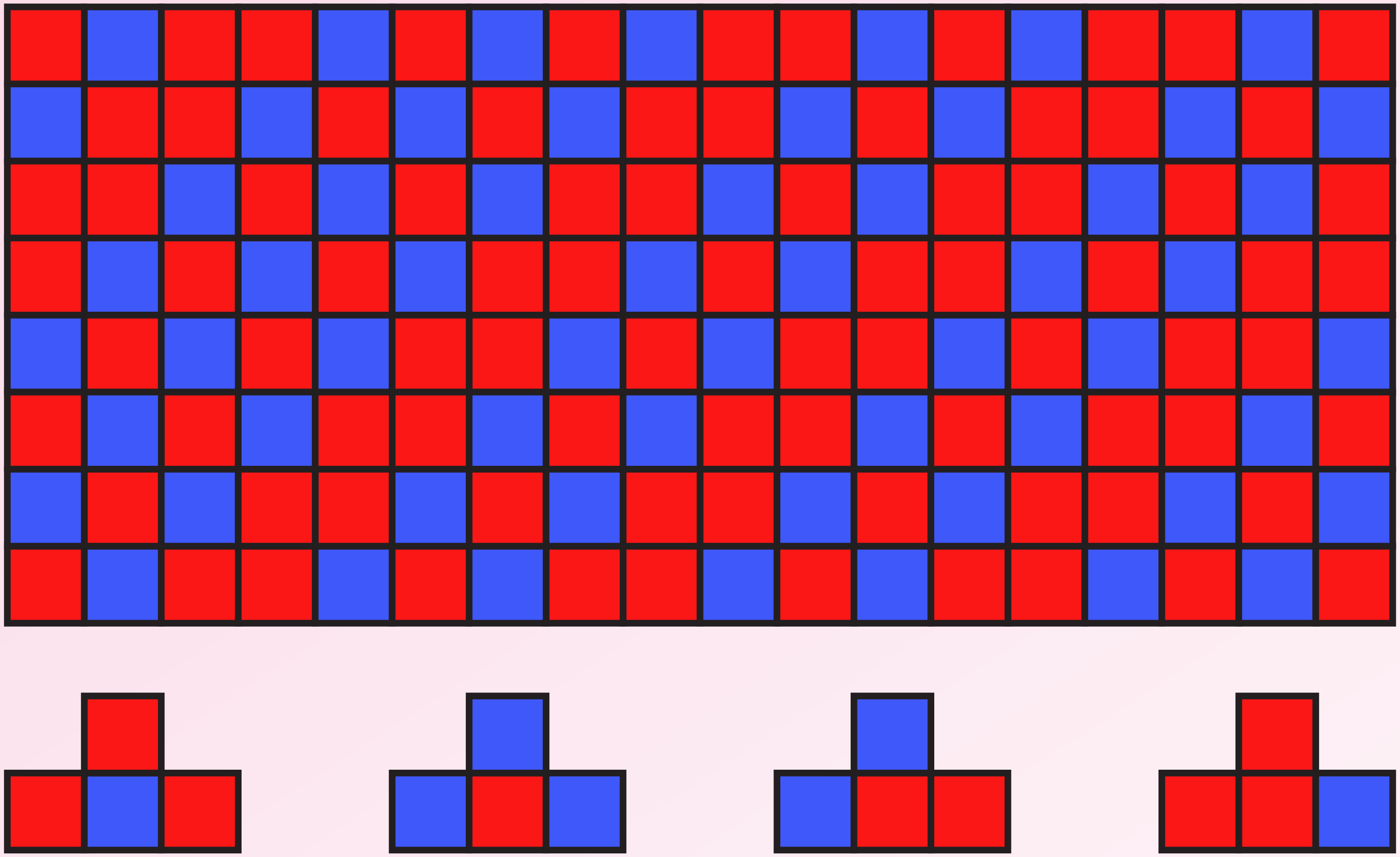


A quantity to measure local complexity: the **pattern complexity**

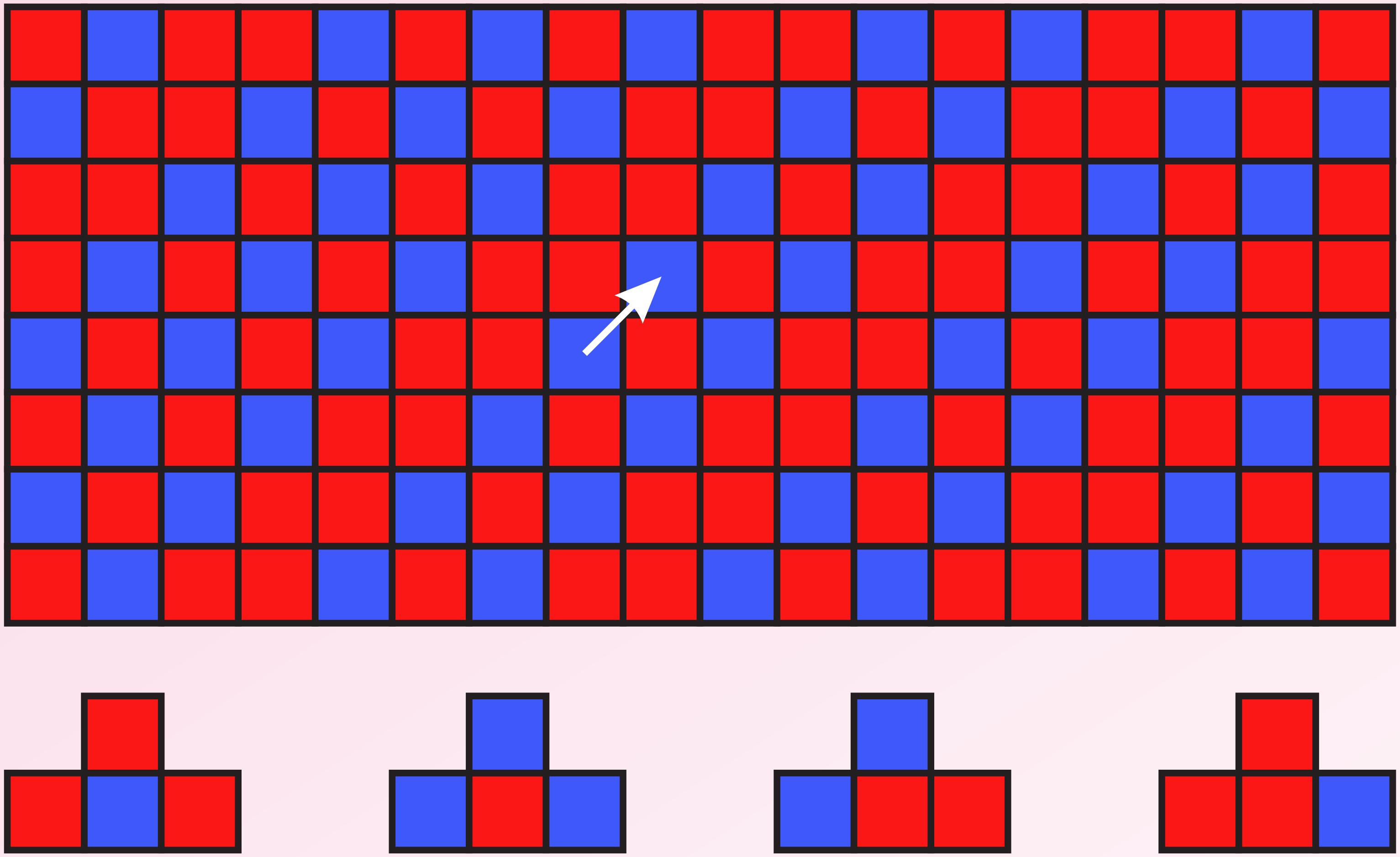
$$P(c, D) = \#\text{ of } D\text{-patterns in } c.$$



If this quantity is small, for some D, global regularities ensue.

The relevant **low complexity threshold**:

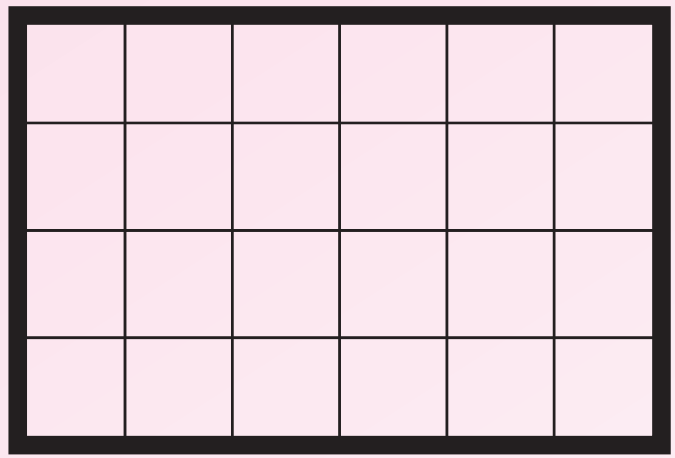
$$P(c, D) \leq |D|$$



Global regularity of interest is periodicity: Configuration is **periodic** if it is invariant under a non-zero translation.

Open problem 1: Nivat's conjecture

Consider $d = 2$ and rectangular D.



Conjecture (Nivat 1997) If $P(c, D) \leq |D|$ for some rectangle D then c is periodic.

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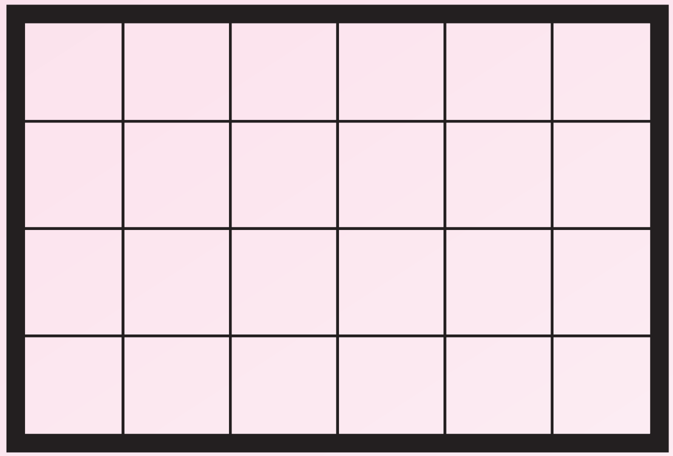

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This would extend the one-dimensional case $d = 1$:

Morse-Hedlund theorem: Let $c \in A^{\mathbb{Z}}$ and $n \in \mathbb{N}$. If c has at most n distinct subwords of length n then c is periodic.

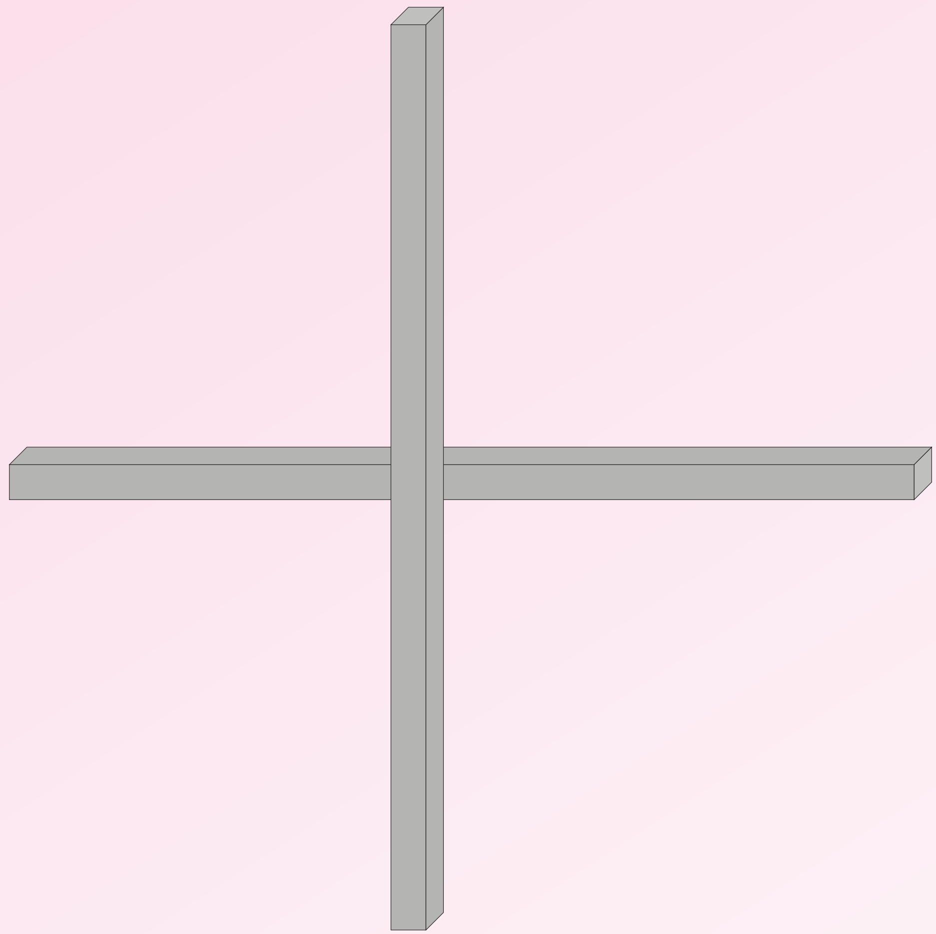
Best known bound in 2D:

Theorem (Cyr, Kra): If $P(c, D) \leq \frac{1}{2}|D|$ for some rectangle D then c is periodic.

Case of narrow rectangles:

Theorem (Cyr, Kra): If D is a rectangle of height at most 3 and $P(c, D) \leq |D|$ then c is periodic.

In 3D and higher dimensional cases the conjecture is false



Non-periodic c

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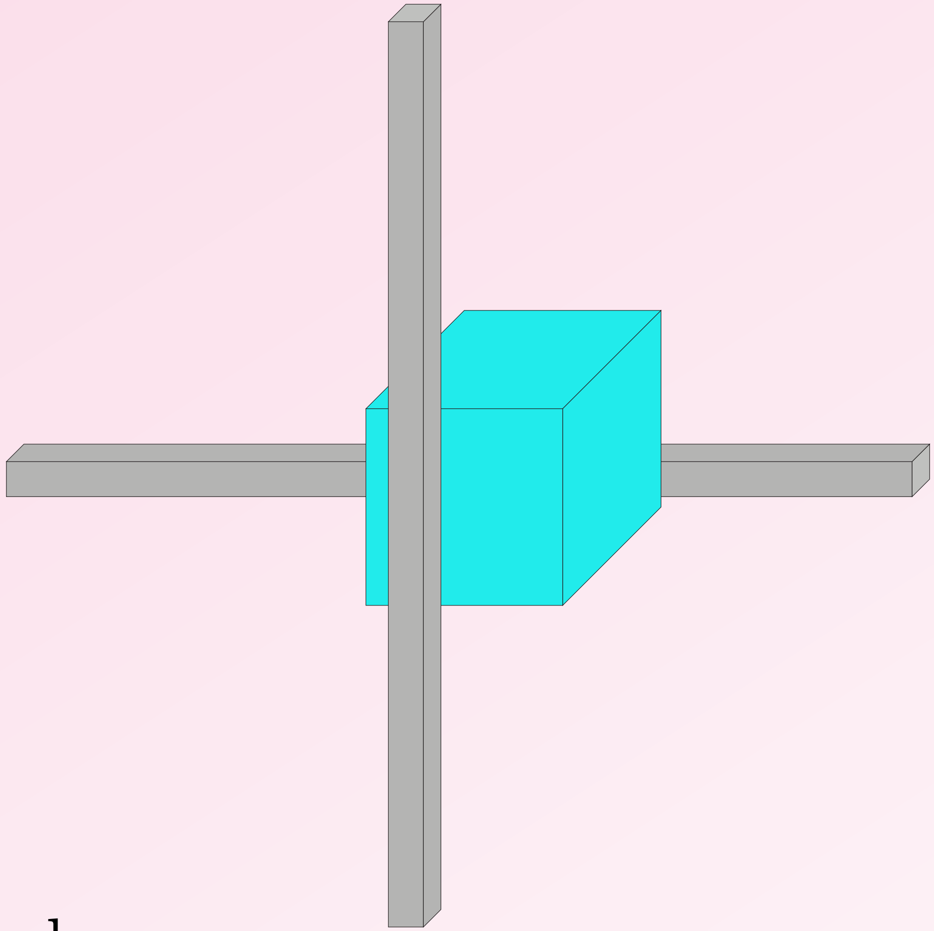

Non-periodic c

D is $n \times n \times n$ cube

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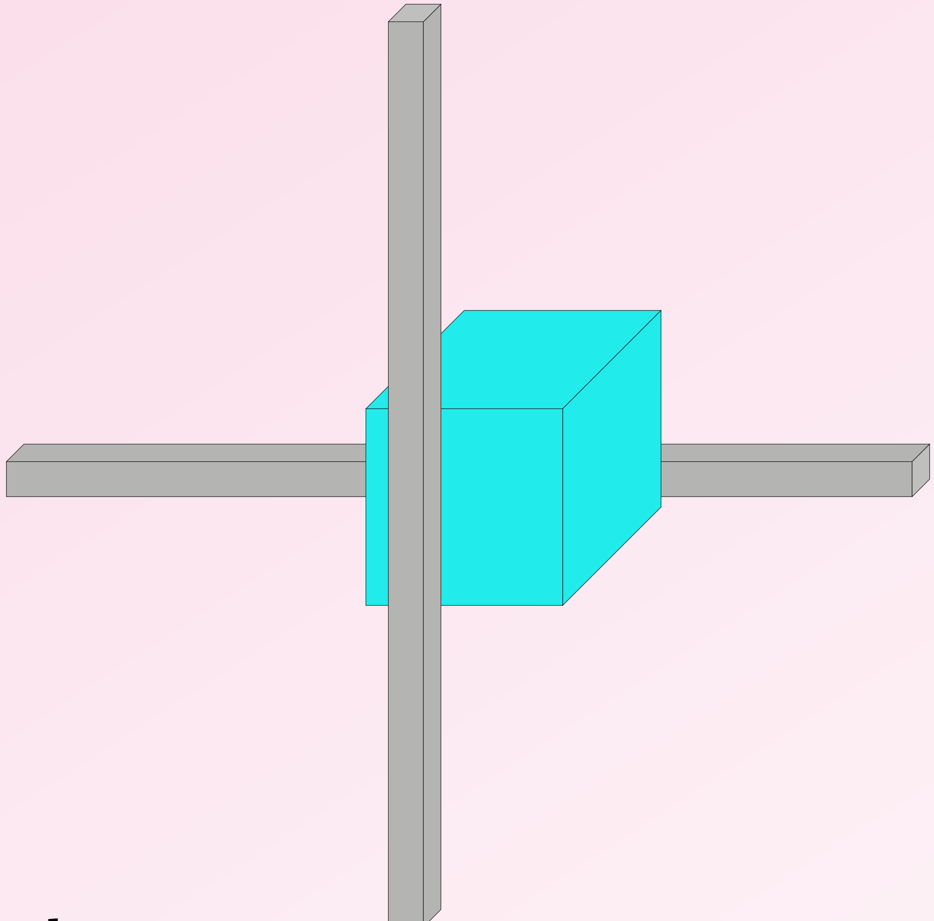

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$P(c, D) = 1 + \ldots$

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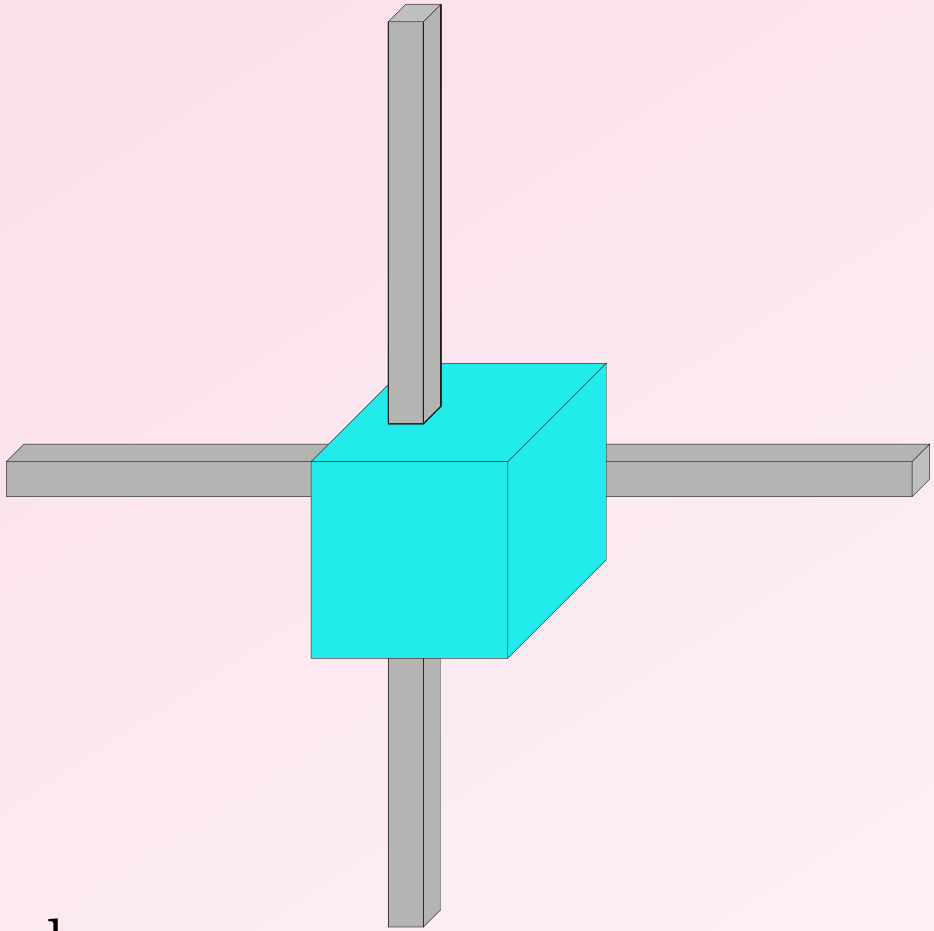

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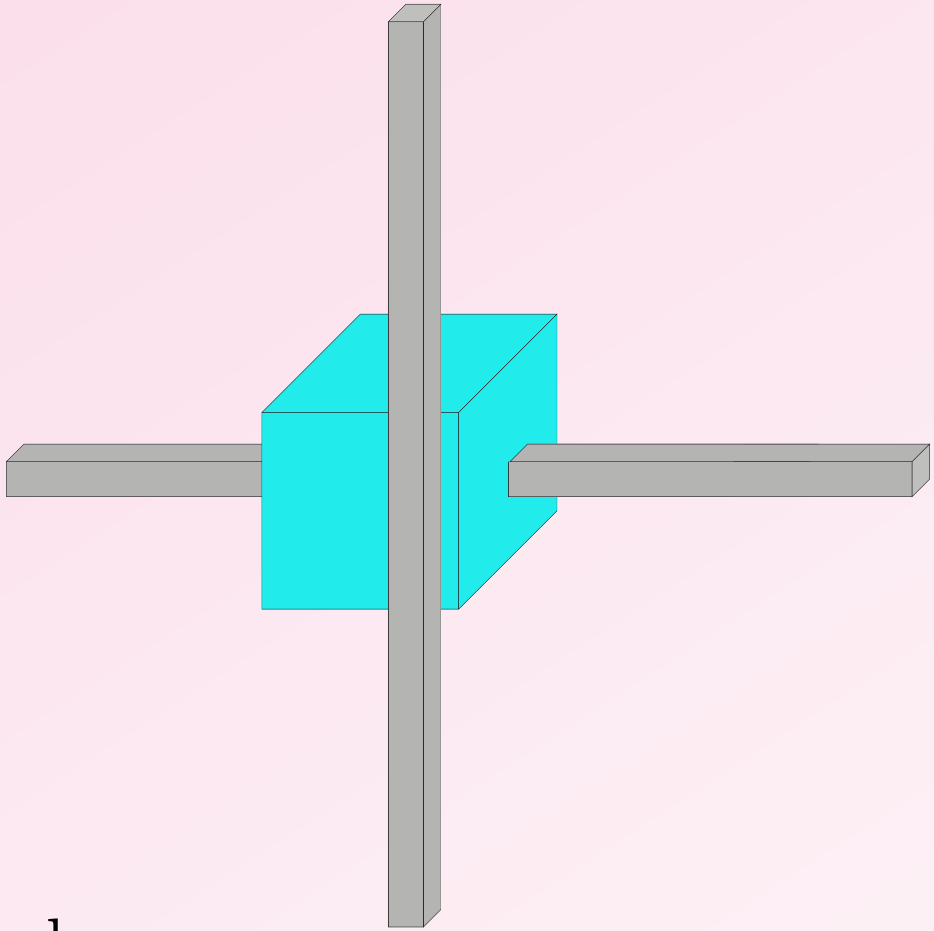

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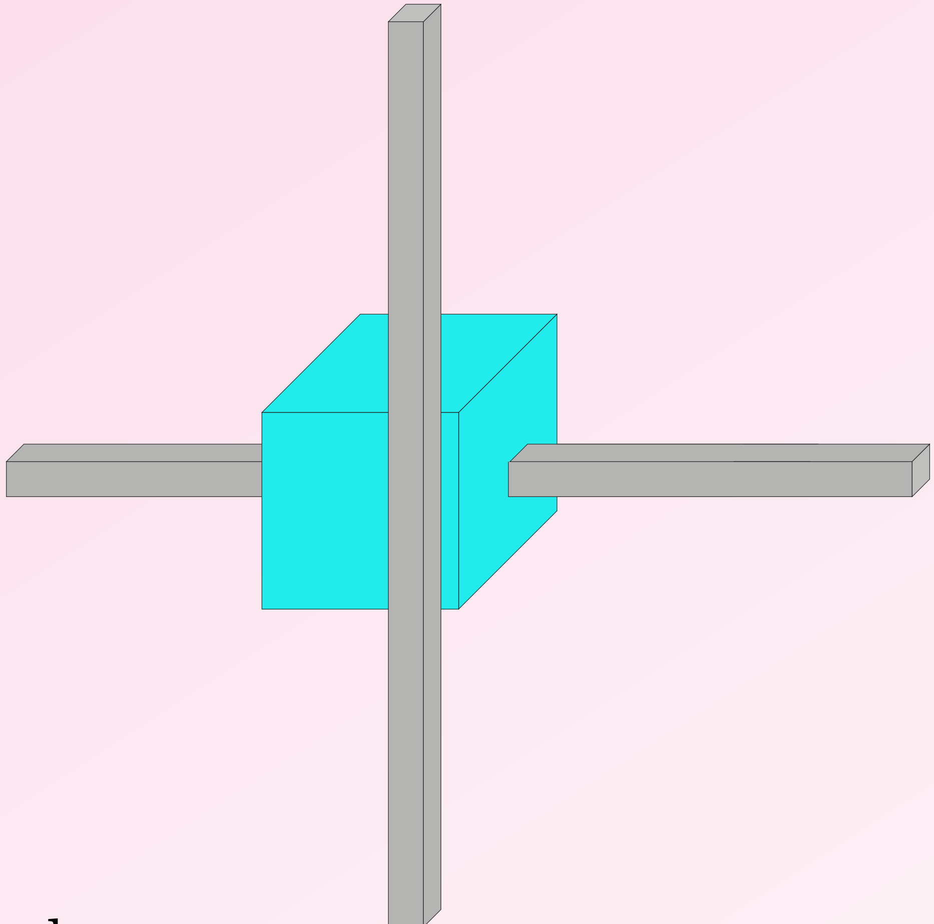

Non-periodic c

D is $n \times n \times n$ cube

$P(c, D) = 1 + n^2 + n^2 < n^3 = |D|$ for large n.

We can prove an asymptotic version in 2D:

Theorem (Kari, Szabados): If $P(c, D) \leq |D|$ for infinitely many different size rectangles D then c is periodic.

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Theorem (Kari, Szabados): If $P(c, D) \leq |D|$ for infinitely many different size rectangles D then c is periodic.

Or stated as **contrapositive:** If c is not periodic then $P(c, D) > |D|$ for all sufficiently large rectangles D.

Open problem 2: Periodic tiling problem

Let $T \subseteq \mathbb{Z}^d$ be finite, and call it a **tile**. A **tiling** is any $C \subseteq \mathbb{Z}^d$ such that

$$C \oplus T = \mathbb{Z}^d.$$

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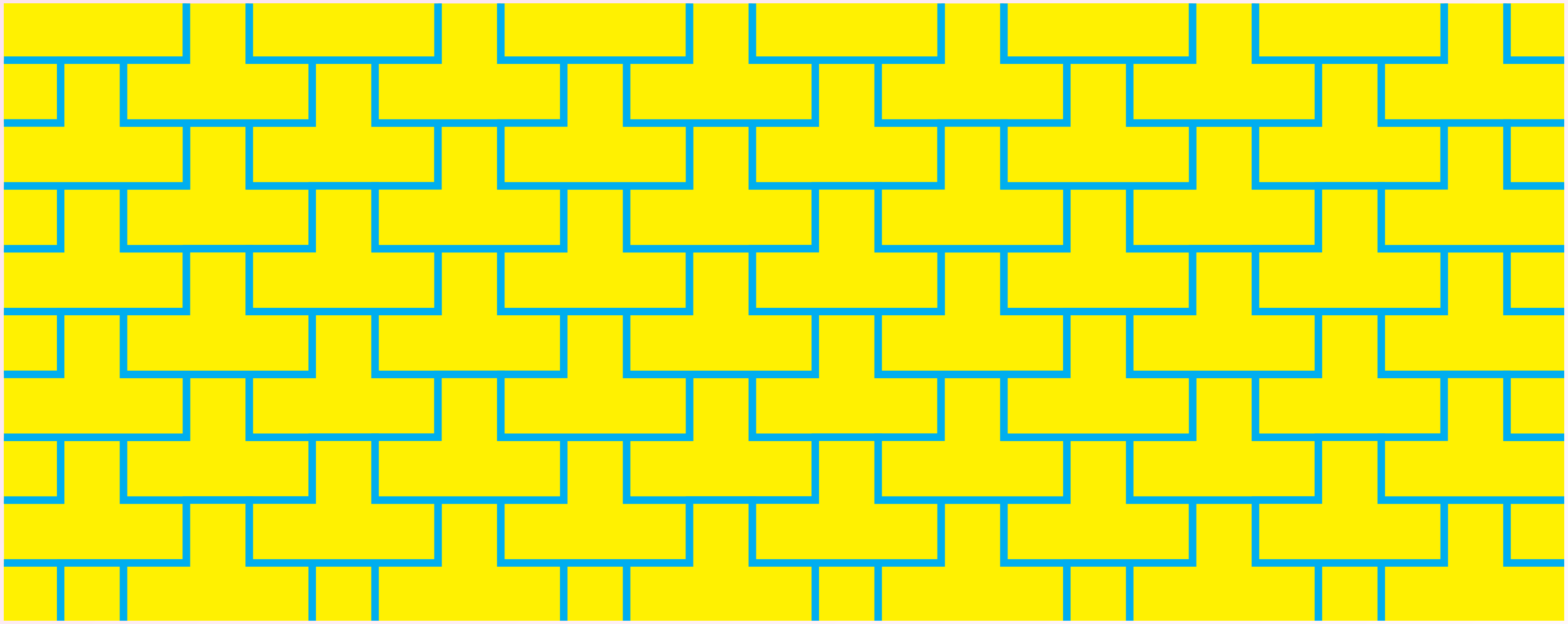
Graphical interpretation: C gives the positions where copies of T are placed to cover $\mathbb{Z}^d$ without gaps or overlaps.



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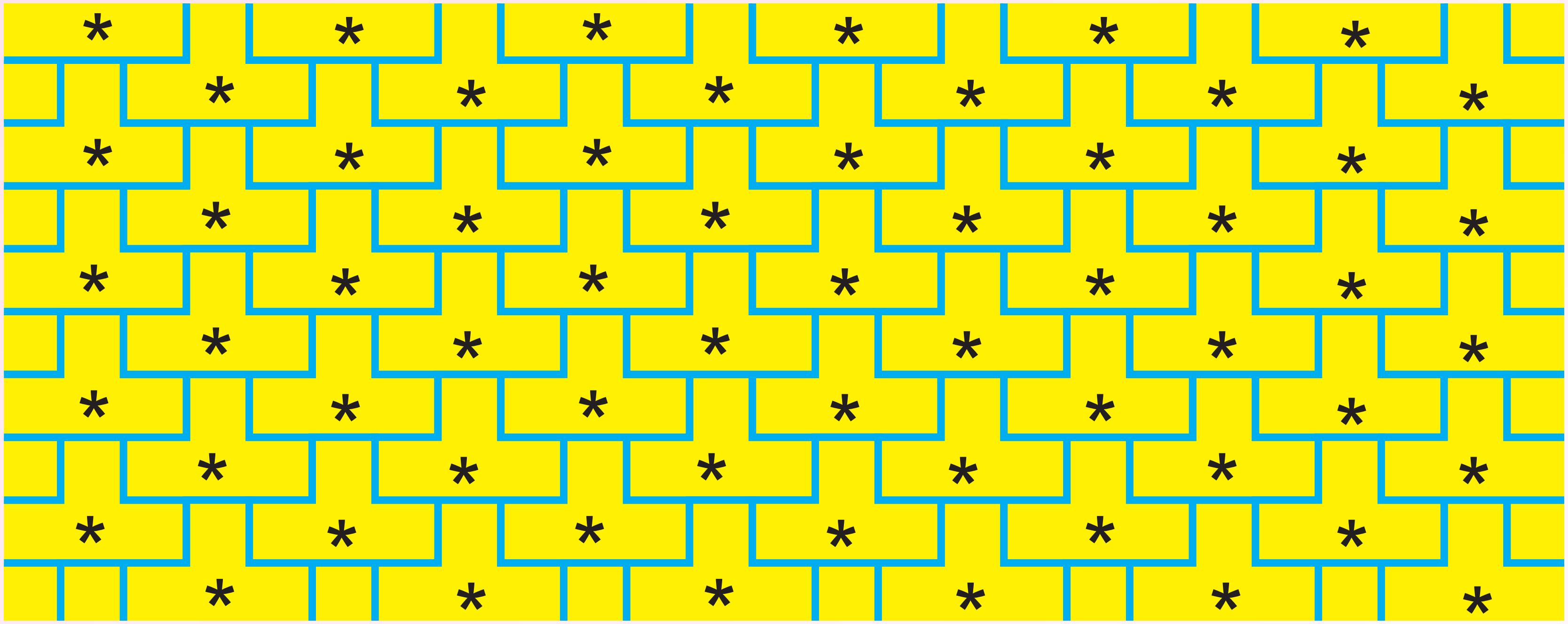

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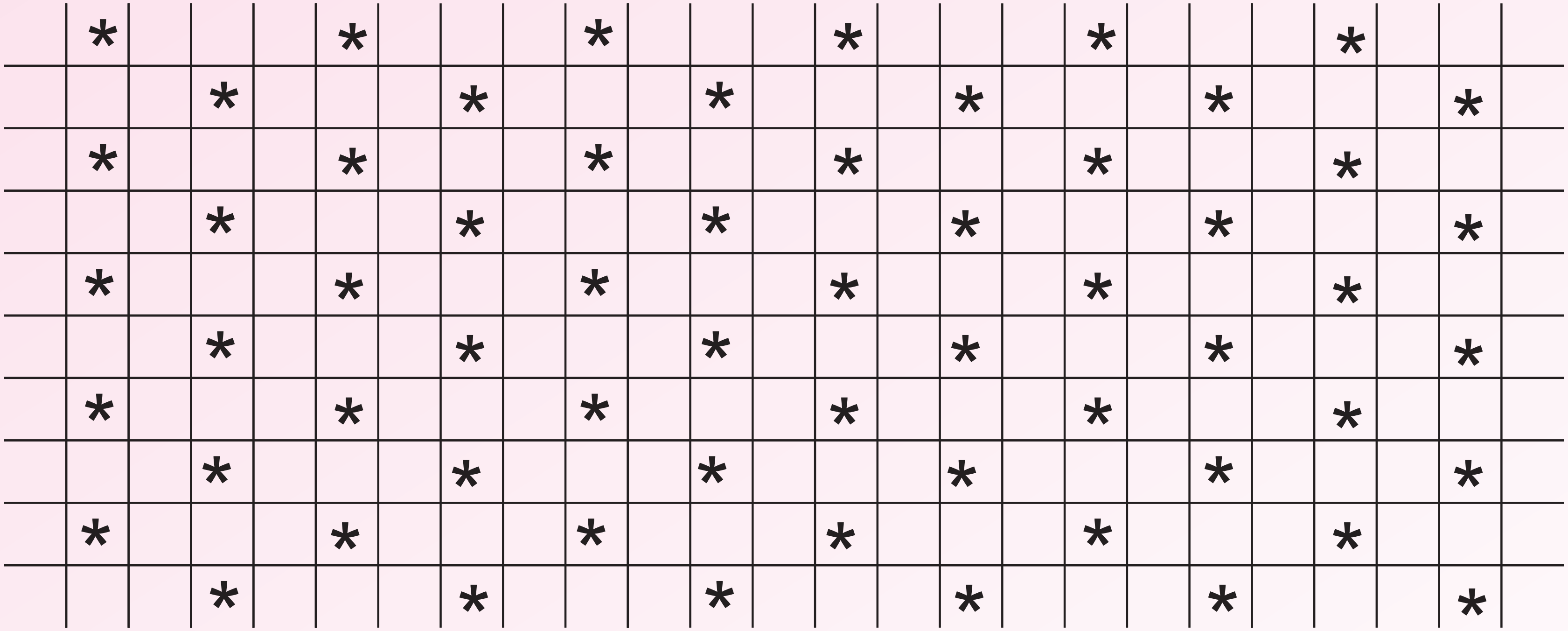
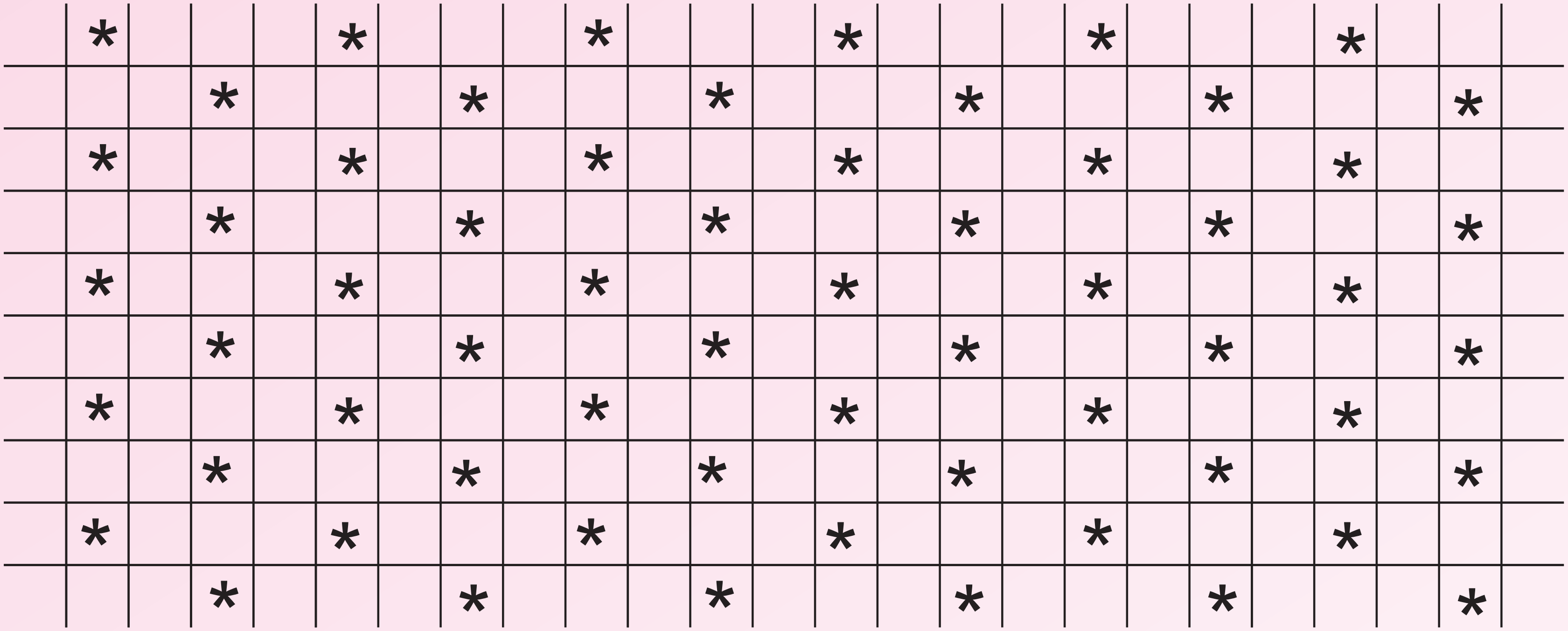
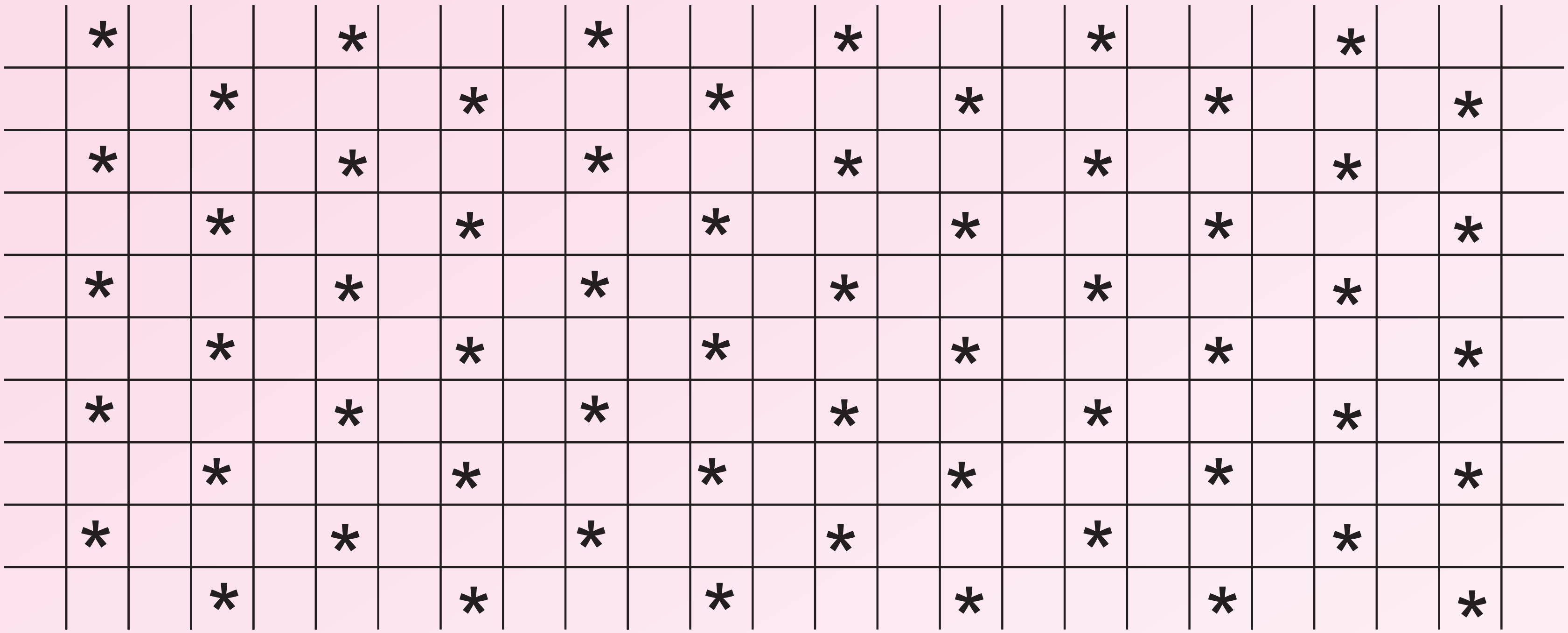
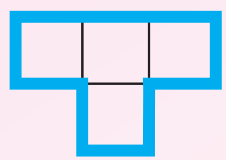




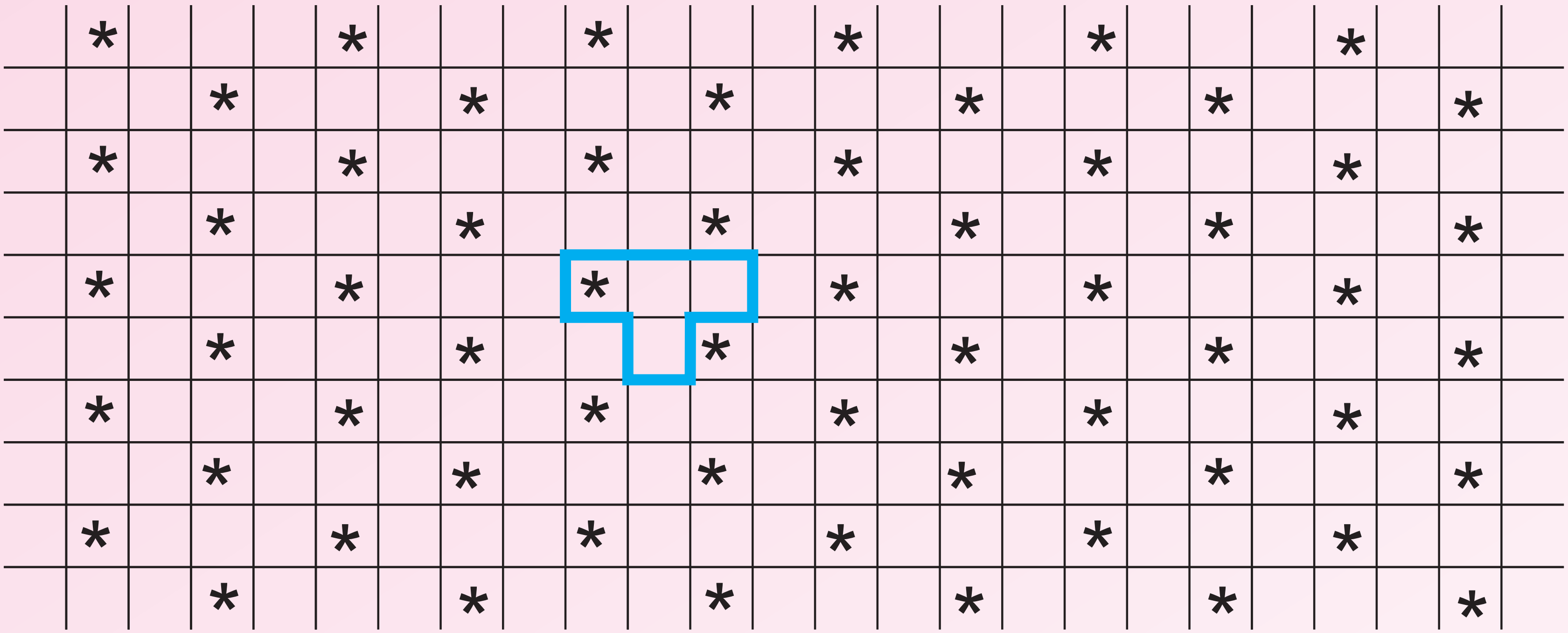
Interpret C as the binary configuration c with

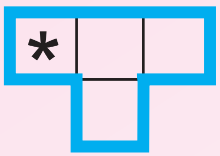
$$c(i) = * \iff i \in C.$$





$(-T)$-patterns of c contain exactly one symbol $*$.





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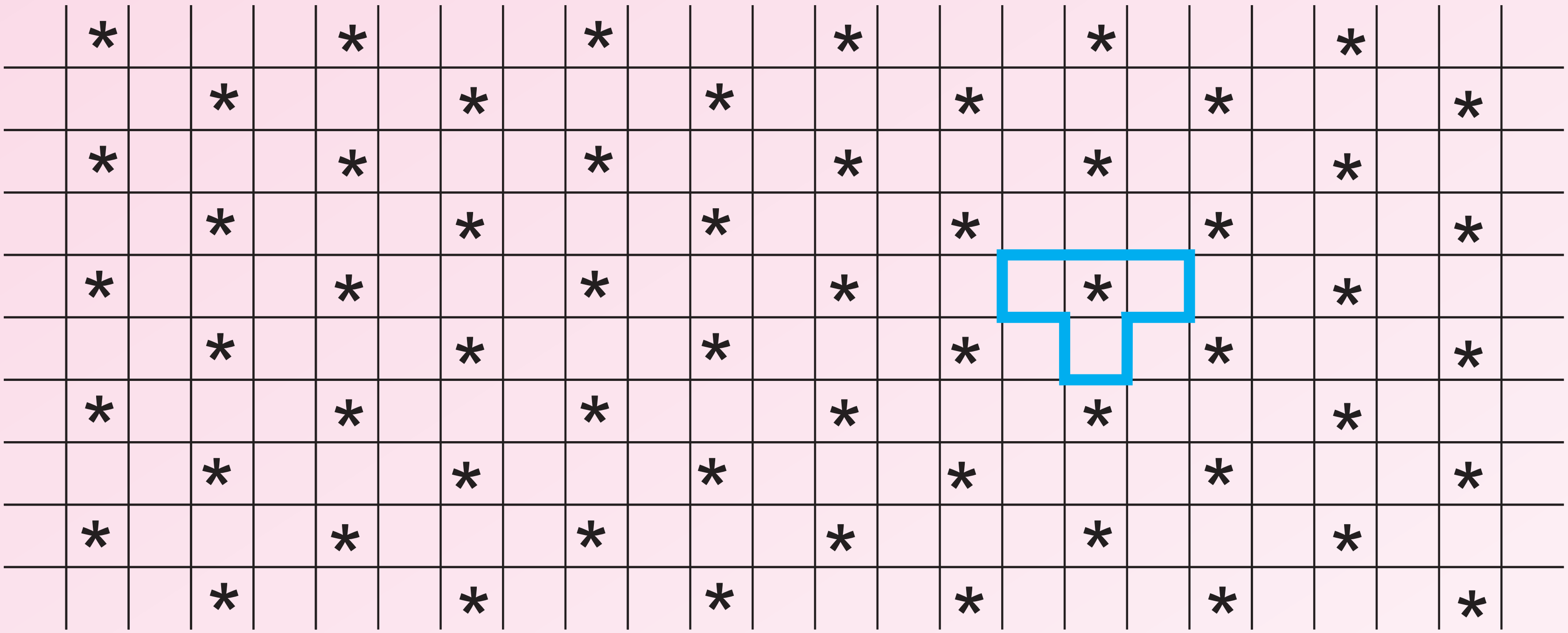

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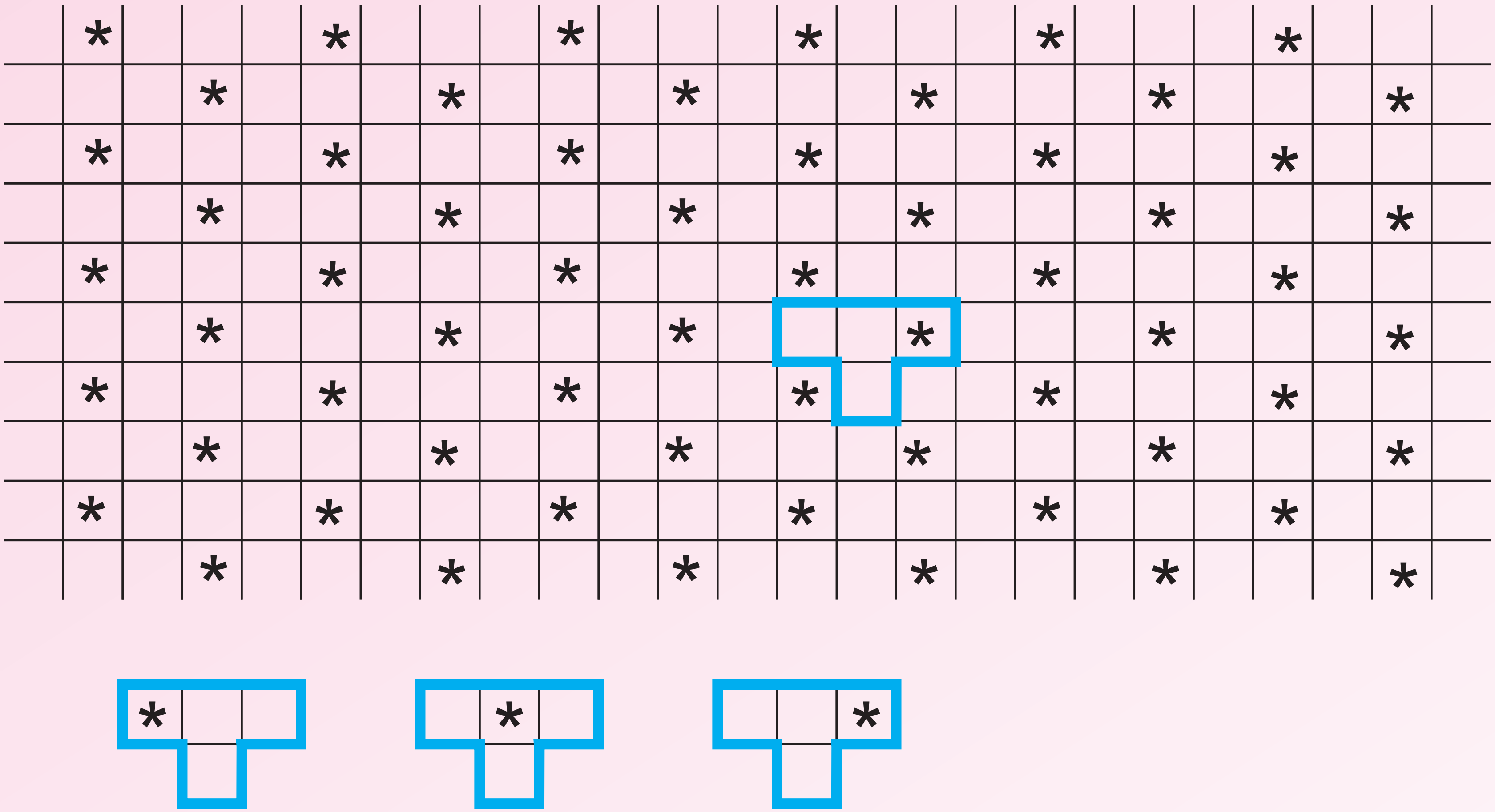

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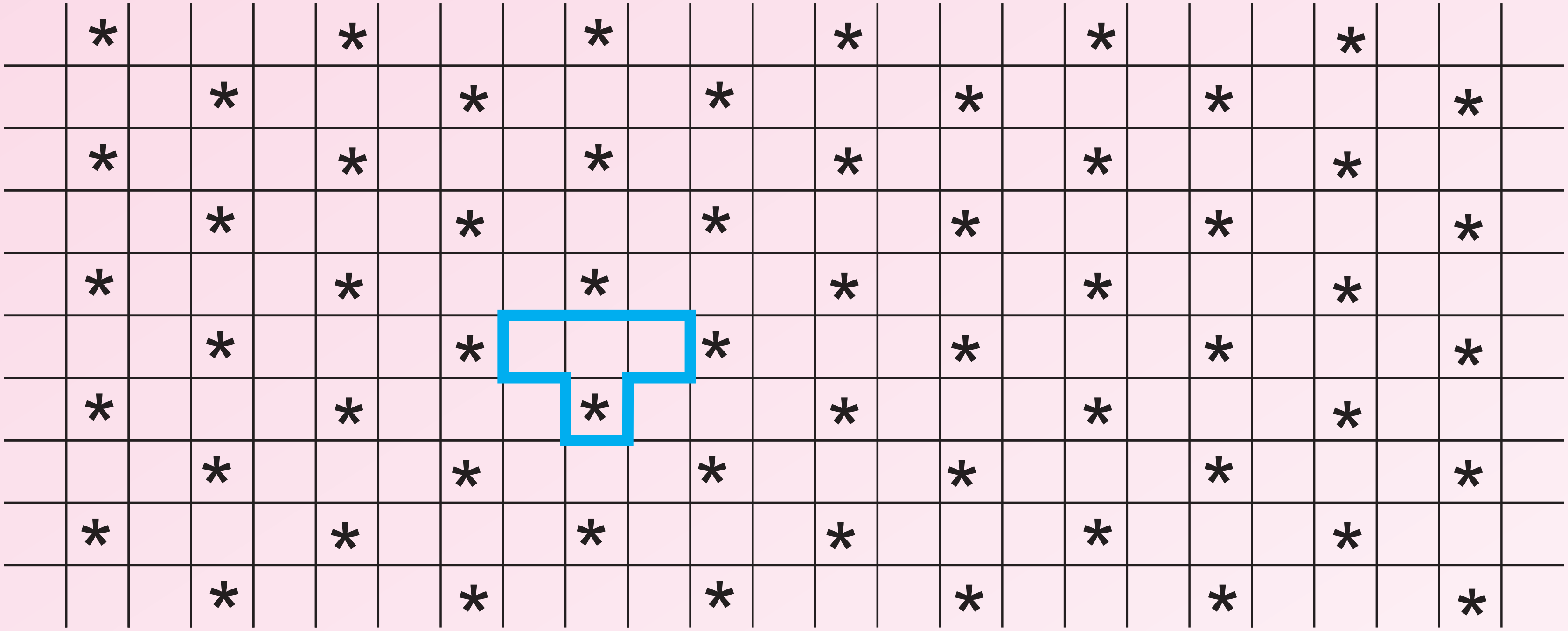

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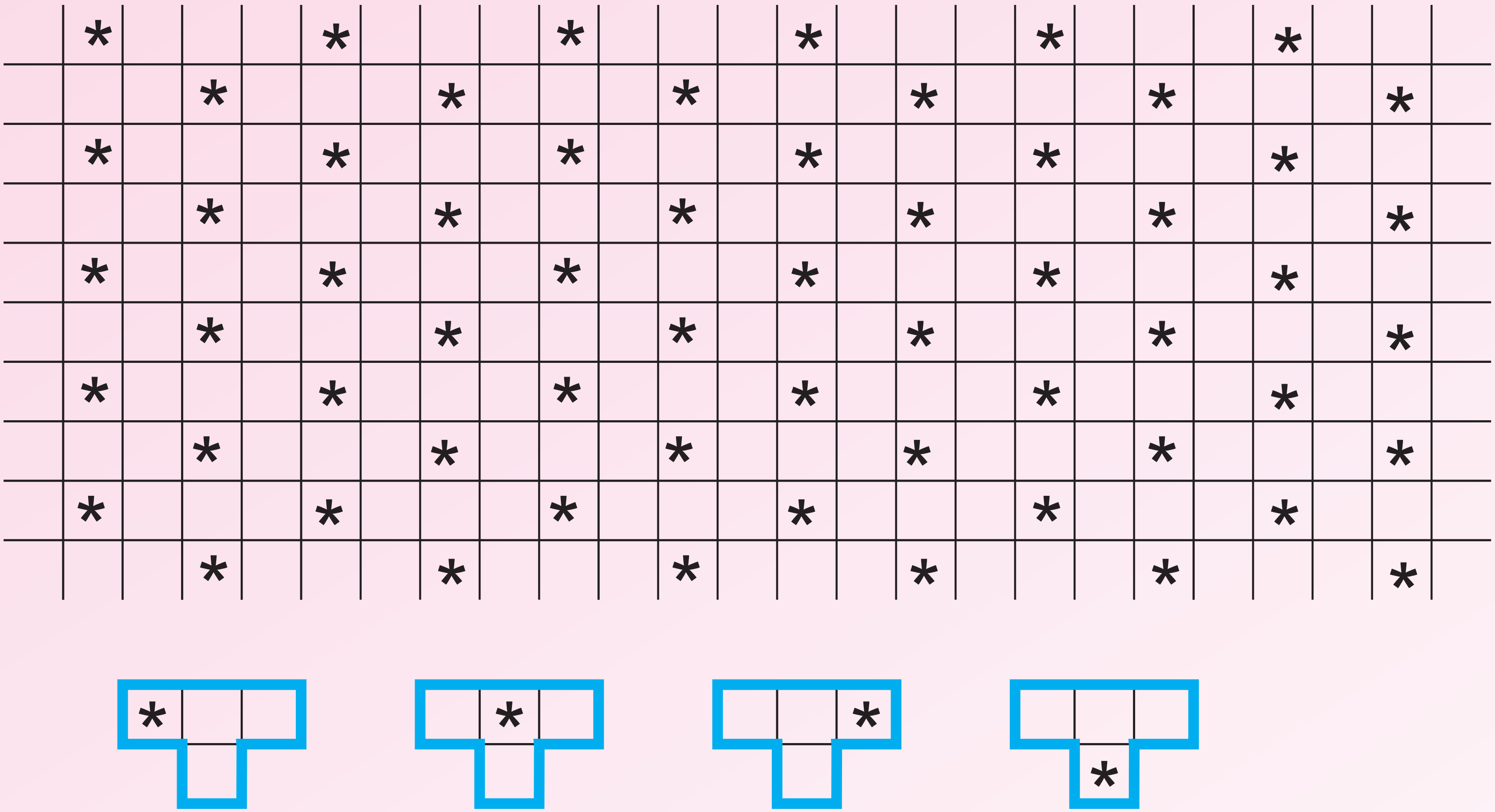
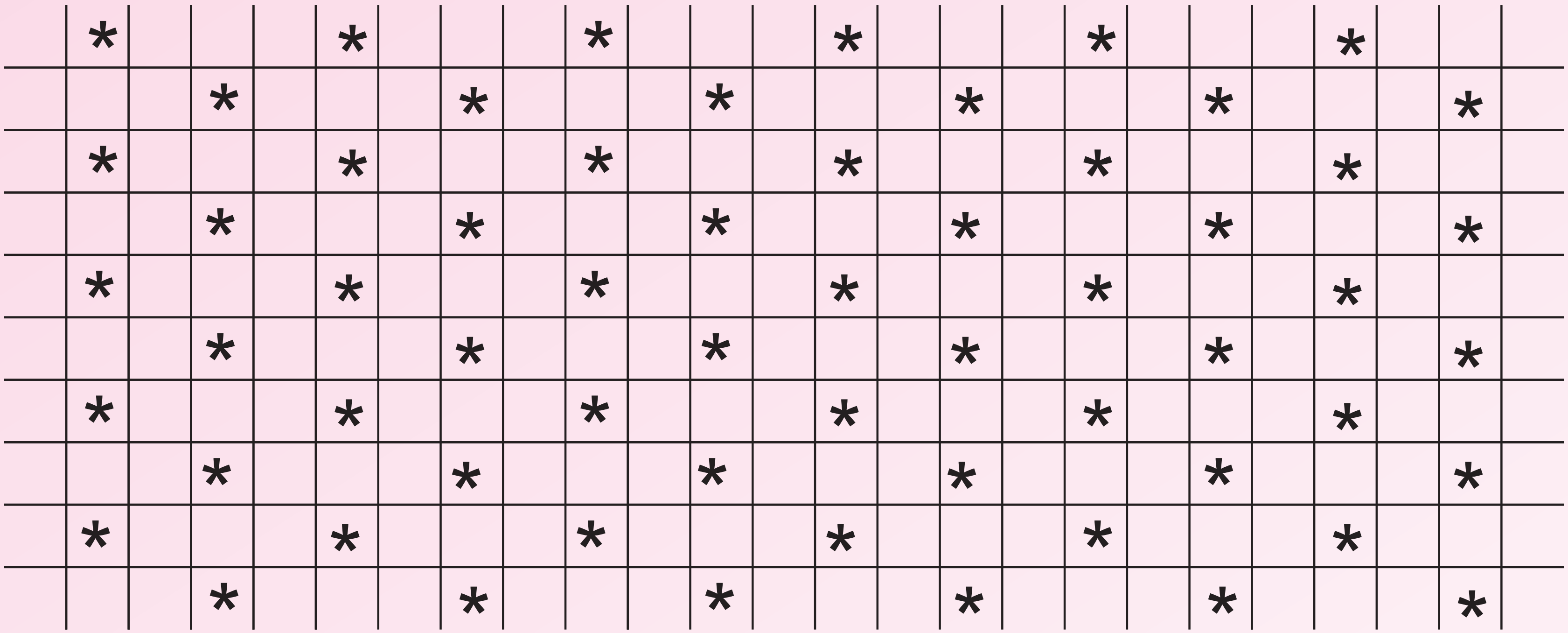


$(-T)$-patterns of c contain exactly one symbol $*$.

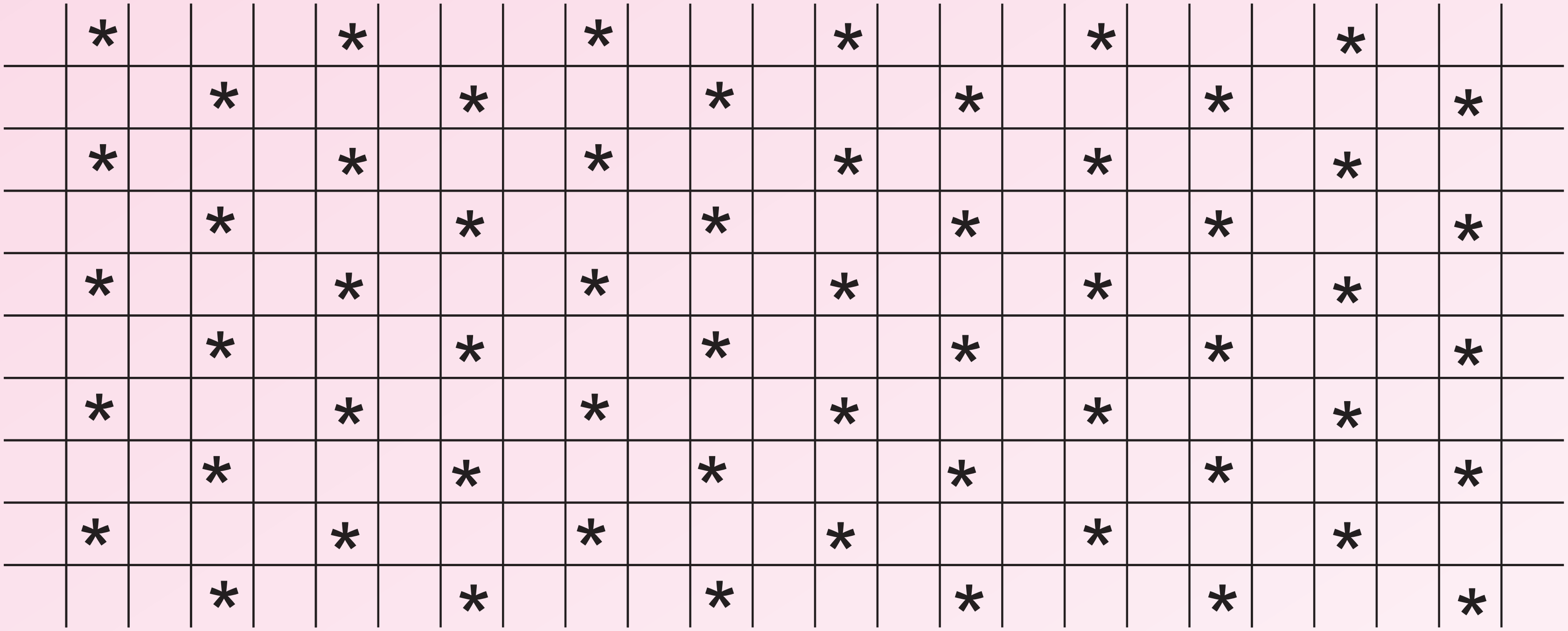
$$P(c, -T) = |-T|$$



$(-T)$-patterns of c contain exactly one symbol $*$.

$$P(c, -T) = |-T|$$

(Also $P(c, T) = |T|$ as any tiling for T is also a tiling for $-T$.)



If X is the **set of all tilings** by T then

$$P(X,T) = |T|$$

where $P(X,T)$ is the number of T-patterns in $c \in X$.

Set X is a low complexity **subshift of finite type (SFT)**.

Periodic tiling problem (Lagarias and Wang 1996): If T admits a tiling C, does it necessarily admit a periodic tiling ?

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Known results:

- Yes if $|T|$ is a prime number (Szegedy 1998).
- Yes in 2D
  - if T is 4-connected (Beauquier and Nivat 1991),
  - in general (Bhattacharya 2016).

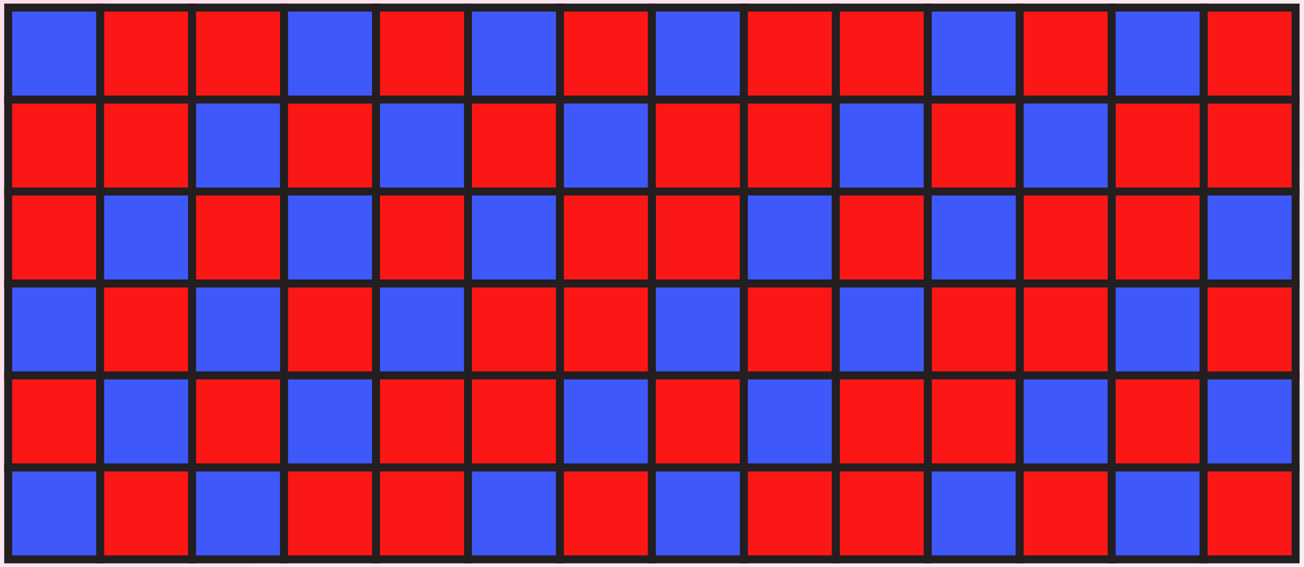
Both the **Nivat's conjecture** and the **Periodic tiling problem** concern periodicity under complexity constraint $P(c, D) \leq |D|$.

We are interested in analogous questions generally.

- **Algorithmic question:** given at most $|D|$ patterns of shape D, does there exist a configuration with only these given D-patterns ? (=emptyness problem of a given low complexity subshift of finite type)

- **Periodicity:** If there exists a configuration whose D-patterns are among the given $\leq |D|$ ones, does there necessarily exist such a configuration that is periodic ?

We study configurations using algebra, so we first replace symbols by integers:



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2	1	1	2	1	2	1	2	1	1	2	1	2	1
1	1	2	1	2	1	2	1	1	2	1	2	1	1
1	2	1	2	1	2	1	1	2	1	2	1	1	2
2	1	2	1	2	1	1	2	1	2	1	1	2	1
1	2	1	2	1	1	2	1	2	1	1	2	1	2
2	1	2	1	1	2	1	2	1	1	2	1	2	1

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2	1	2	1	2	1	1	2	1	2	1	1	2	1
1	2	1	2	1	1	2	1	2	1	1	2	1	2
2	1	2	1	1	2	1	2	1	1	2	1	2	1

2	1	1	2	1	2	1	2	1	1	2	1	2	1
1	1	2	1	2	1	2	1	1	2	1	2	1	1
1	2	1	2	1	2	1	1	2	1	2	1	1	2
2	1	2	1	2	1	1	2	1	2	1	1	2	1
1	2	1	2	1	1	2	1	2	1	1	2	1	2
2	1	2	1	1	2	1	2	1	1	2	1	2	1

D-patterns are viewed as $|D|$-dimensional numerical vectors.

2	1	1	2	1	2	1	2	1	1	2	1	2	1
1	1	2	1	2	1	2	1	1	2	1	2	1	1
1	2	1	2	1	2	1	1	2	1	2	1	1	2
2	1	2	1	2	1	1	2	1	2	1	1	2	1
1	2	1	2	1	1	2	1	2	1	1	2	1	2
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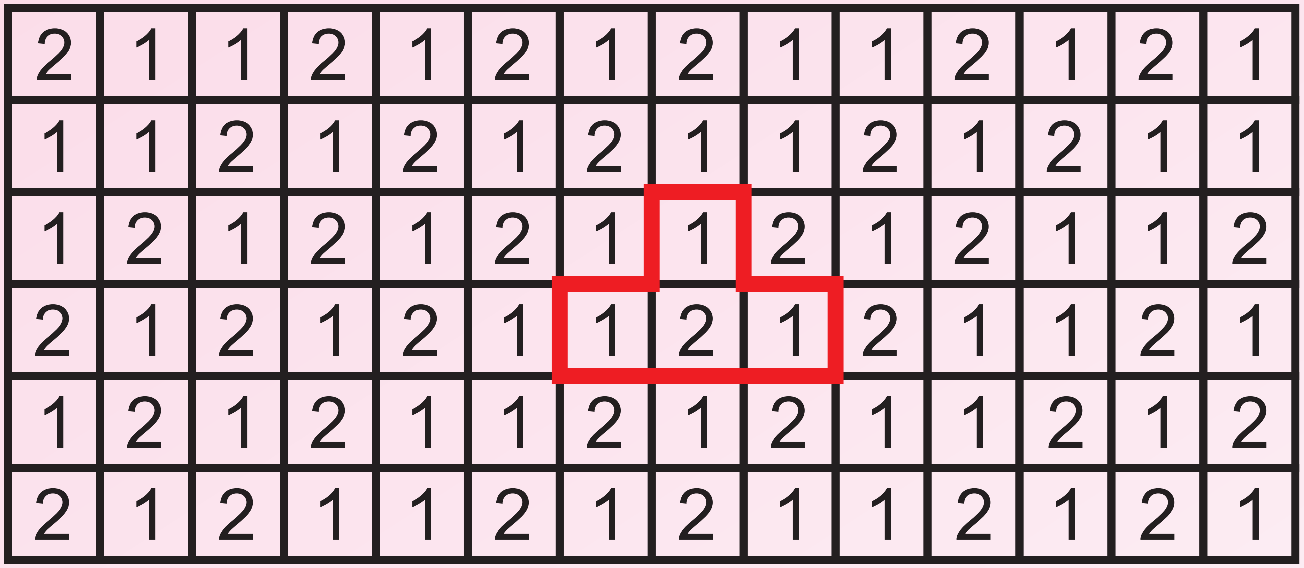
$$(1, 1, 1, 2)$$



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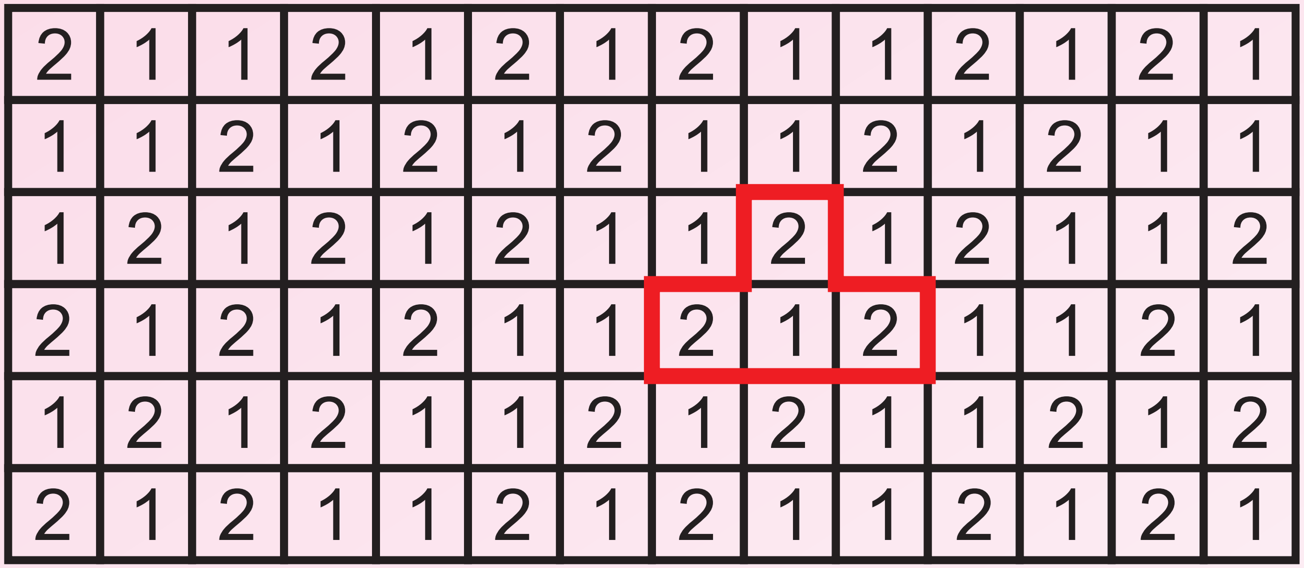

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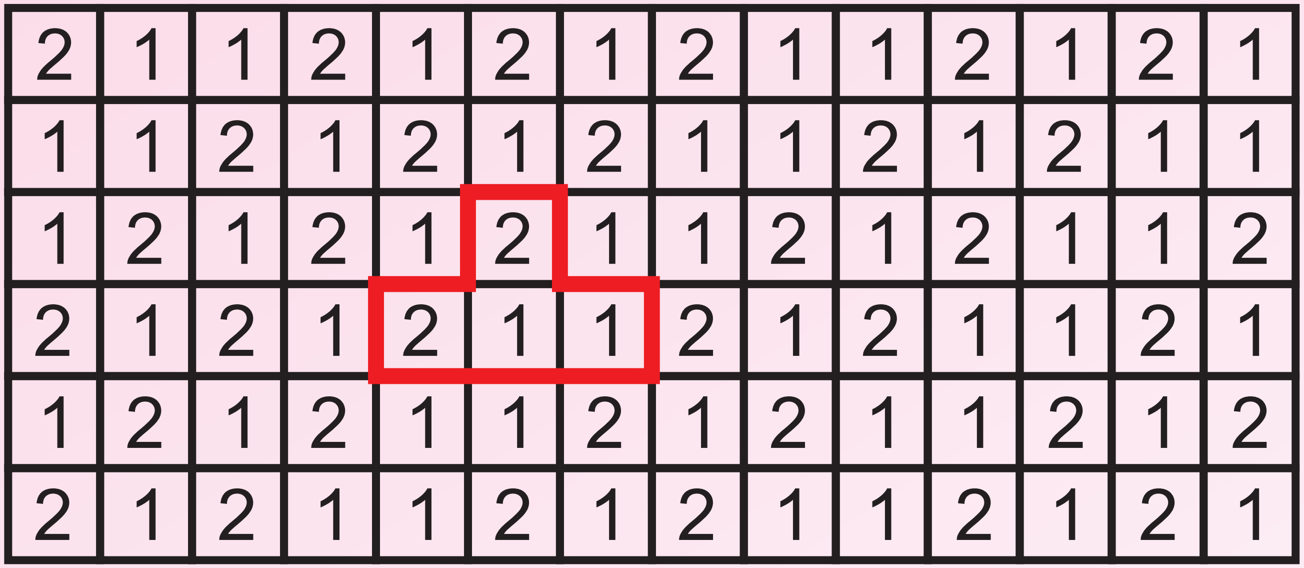

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2	1	1	2	1	2	1	2	1	1	2	1	2	1
1	1	2	1	2	1	2	1	1	2	1	2	1	1
1	2	1	2	1	2	1	1	2	1	2	1	1	2
2	1	2	1	2	1	1	2	1	2	1	1	2	1
1	2	1	2	1	1	2	1	2	1	1	2	1	2
2	1	2	1	1	2	1	2	1	1	2	1	2	1

- If $P(c, D) < |D|$ then there is an (integer) vector orthogonal to all D-patterns of c.

Indeed: the number $P(c, D)$ of distinct vectors is less than the dimension $|D|$ of the linear space.

2	1	1	2	1	2	1	2	1	1	2	1	2	1
1	1	2	1	2	1	2	1	1	2	1	2	1	1
1	2	1	2	1	2	1	1	2	1	2	1	1	2
2	1	2	1	2	1	1	2	1	2	1	1	2	1
1	2	1	2	1	1	2	1	2	1	1	2	1	2
2	1	2	1	1	2	1	2	1	1	2	1	2	1

- If $P(c, D) < |D|$ then there is an (integer) vector orthogonal to all D-patterns of c.
- Even if $P(c, D) = |D|$ we can add a suitable rational constant to c to make the vectors linearly dependent. Also then an orthogonal vector exists.

2	1	1	2	1	2	1	2	1	1	2	1	2	1
1	1	2	1	2	1	2	1	1	2	1	2	1	1
1	2	1	2	1	2	1	1	2	1	2	1	1	2
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- If $P(c, D) < |D|$ then there is an (integer) vector orthogonal to all D-patterns of c.
- Even if $P(c, D) = |D|$ we can add a suitable rational constant to c to make the vectors linearly dependent. Also then an orthogonal vector exists.

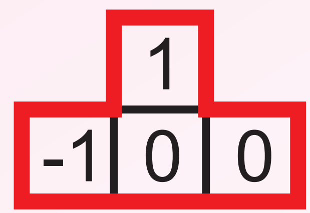
This is OK: we are free to choose the numerical encoding.

2	1	1	2	1	2	1	2	1	1	2	1	2	1
1	1	2	1	2	1	2	1	1	2	1	2	1	1
1	2	1	2	1	2	1	1	2	1	2	1	1	2
2	1	2	1	2	1	1	2	1	2	1	1	2	1
1	2	1	2	1	1	2	1	2	1	1	2	1	2
2	1	2	1	1	2	1	2	1	1	2	1	2	1

$$\left.\begin{array}{c}(1,1,1,2)\\(1,1,2,1)\\(2,2,1,2)\\(2,2,1,1)\end{array}\right\} \quad \perp \quad (1,-1,0,0)$$

2	1	1	2	1	2	1	2	1	1	2	1	2	1
1	1	2	1	2	1	2	1	1	2	1	2	1	1
1	2	1	2	1	2	1	1	2	1	2	1	1	2
2	1	2	1	2	1	1	2	1	2	1	1	2	1
1	2	1	2	1	1	2	1	2	1	1	2	1	2
2	1	2	1	1	2	1	2	1	1	2	1	2	1

$$\left.\begin{array}{c}(1,1,1,2)\\(1,1,2,1)\\(2,2,1,2)\\(2,2,1,1)\end{array}\right\} \quad \perp \quad (1,-1,0,0)$$



The orthogonal vector is a **filter** whose convolution with c is the zero configuration. We say it **annihilates** configuration c.

2	1	1	2	1	2	1	2	1	1	2	1	2	1
1	1	2	1	2	1	2	1	1	2	1	2	1	1
1	2	1	2	1	2	1	1	2	1	2	1	1	2
2	1	2	1	2	1	1	2	1	2	1	1	2	1
1	2	1	2	1	1	2	1	2	1	1	2	1	2
2	1	2	1	1	2	1	2	1	1	2	1	2	1

Conclusion: If $P(c, D) \leq |D|$ then symbols can be represented as integers in such a way that some non-trivial integer filter annihilates c.

To use algebraic geometry, we next represent c as a **power series** (negative exponents included).

2	1	2	1	1
1	2	1	1	2
2	1	1	2	1
1	1	2	1	2
1	2	1	2	1

$$c \longleftrightarrow \sum_{(i_1,\ldots,i_d)\in\mathbb{Z}^d} c(i_1,\ldots,i_d)x_1^{i_1}\ldots x_d^{i_d}$$

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$2x^{-3}y^{2}$	$x^{-3}y^{2}$	$2x^{0}y^{2}$	$x^{1}y^{2}$	$x^{2}y^{2}$
$x^{-2}y^{1}$	$2x^{-1}y^{1}$	$x^{0}y^{1}$	$x^{1}y^{1}$	$2x^{2}y^{1}$
$2x^{-2}y^{0}$	$x^{-1}y^{0}$	$x^{0}y^{0}$	$2x^{1}y^{0}$	$x^{2}y^{0}$
$x^{-2}y^{-1}$	$x^{-1}y^{-1}$	$2x^{0}y^{-1}$	$x^{1}y^{-1}$	$2x^{2}y^{-1}$
$x^{-2}y^{-2}$	$2x^{-1}y^{-2}$	$x^{0}y^{-2}$	$2x^{1}y^{-2}$	$x^{2}y^{-2}$

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$$\ldots + 2x^{-3}y^{2} + x^{-3}y^{2} + 2x^{0}y^{2} + x^{1}y^{2} + x^{2}y^{2} + \ldots$$

$$\ldots + x^{-2}y^{1} + 2x^{-1}y^{1} + x^{0}y^{1} + x^{1}y^{1} + 2x^{2}y^{1} + \ldots$$

$$\ldots + 2x^{-2}y^{0} + x^{-1}y^{0} + x^{0}y^{0} + 2x^{1}y^{0} + x^{2}y^{0} + \ldots$$

$$\ldots + x^{-2}y^{-1} + x^{-1}y^{-1} + 2x^{0}y^{-1} + x^{1}y^{-1} + 2x^{2}y^{-1} + \ldots$$

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Notations:

- $X = (x_1, \ldots, x_d)$
- For $I = (i_1, \ldots, i_d) \in \mathbb{Z}^d$ we denote by

  $$X^I = x_1^{i_1} \ldots x_d^{i_d}$$

  the monomial that represents cell I.

$$c \longleftrightarrow \sum_{(i_1,\ldots,i_d)\in\mathbb{Z}^d} c(i_1,\ldots,i_d)x_1^{i_1}\ldots x_d^{i_d} = \underbrace{\sum_{I\in\mathbb{Z}^d} c(I)X^I}_{c(X)}$$

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$$c(X) = \sum_{I \in \mathbb{Z}^d} c(I) X^I$$

The configuration is now a power series $c(X)$ that is

- **integral** (=all coefficients are integers), and
- **finitary** (=finite number of distinct coefficients)

$$\boxed{c(X) = \sum_{I \in \mathbb{Z}^d} c(I) X^I}$$

Multiplying $c(X)$ by monomial X^J gives **translation** by $J \in \mathbb{Z}^d$:

$$X^J \cdot c(X) = \sum_{I \in \mathbb{Z}^d} c(I) X^{I+J}$$

So $c(X)$ is ***J*-periodic** if and only if $X^J \cdot c(X) = c(X)$, i.e.,

$$(X^J - 1) c(X) = 0$$

$$c(X) = \sum_{I \in \mathbb{Z}^d} c(I) X^I$$

Multiplying $c(X)$ by a (Laurent) polynomial $f(X)$ is a convolution, corresponding to **filtering** operation.

We say that $f(X)$ **annihilates** $c(X)$ if $f(X)c(X) = 0$.

$$c(X) = \sum_{I \in \mathbb{Z}^d} c(I) X^I$$

- Zero polynomial $f(X) = 0$ annihilates every configuration – it is the **trivial annihilator**.
- Binomial $X^I - 1$ annihilates $c(X)$ if and only if $c(X)$ is I-periodic.
- Annihilators of $c(X)$ form an **ideal**:
  - if $f(X)$ and $g(X)$ annihilate $c(X)$, also $f(X) + g(X)$ annihilates it,
  - if $f(X)$ annihilates $c(X)$ then also $g(X)f(X)$ annihilates it, for all $g(X)$.

Define

$$\mathrm{Ann}(c) = \{f(X) \in \mathbb{C}[X] \mid f(X)c(X) = 0\}.$$

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Remarks:

- We consider polynomials (not Laurent polynomials!) so that we can directly rely on polynomial algebra. **No problem:** any Laurent polynomial annihilator can be made into a proper polynomial annihilator by multiplying it with suitable monomial X^I.
- We allow complex coefficients because we need algebraicly closed field to apply Hilbert's Nullstellensatz.
- $\mathrm{Ann}(c)$ is indeed an ideal of the polynomial ring $\mathbb{C}[X]$.

$$\mathrm{Ann}(c) = \{f(X) \in \mathbb{C}[X] \mid f(X)c(X) = 0\}$$

Our setup (=low complexity configuration) is an integral, finitary $c(X)$ that has some non-trivial **integral annihilator**

$$f(X) \in \mathrm{Ann}(c) \cap \mathbb{Z}[X]$$

$$\boxed{\operatorname{Ann}(c) = \{f(X) \in \mathbb{C}[X] \mid f(X)c(X) = 0\}}$$

To prove that $\operatorname{Ann}(c)$ contains "simple" polynomials we use

Nullstellensatz (Hilbert): Let $g(X)$ be a polynomial. Suppose that $g(Z) = 0$ for all Z in the **variety**

$$\{Z \in \mathbb{C}^d \mid \ f(Z) = 0 \text{ for all } f \in \operatorname{Ann}(c) \ \}.$$

Then $g^k \in \operatorname{Ann}(c)$ for some $k \in \mathbb{N}$.

In the following:

$c(X)$ a finitary, integral power series

$f(X) = \sum_{I \in \mathcal{I}} a_I X^I$ its non-trivial integral annihilator polynomial

($\mathcal{I}$ is the support: $a_I \neq 0$ for all $I \in \mathcal{I}$)

Lemma: $f(X^n) \in \mathrm{Ann}(c)$ for every $n \in \mathbb{N}$ whose prime factors are sufficiently large.

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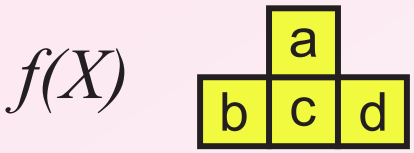

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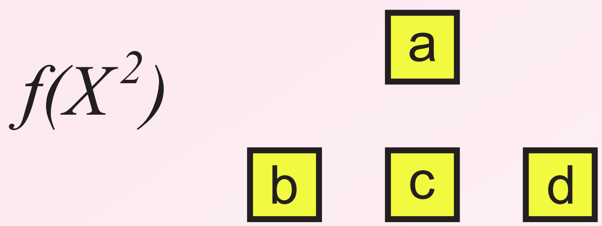

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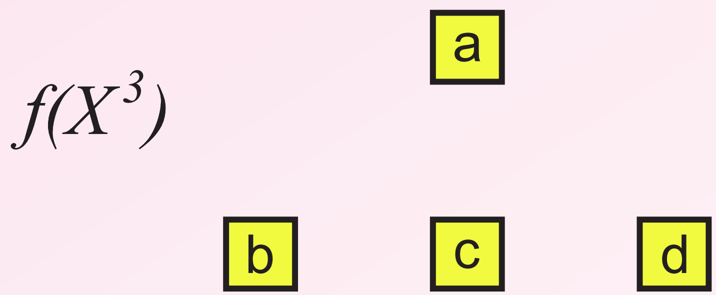

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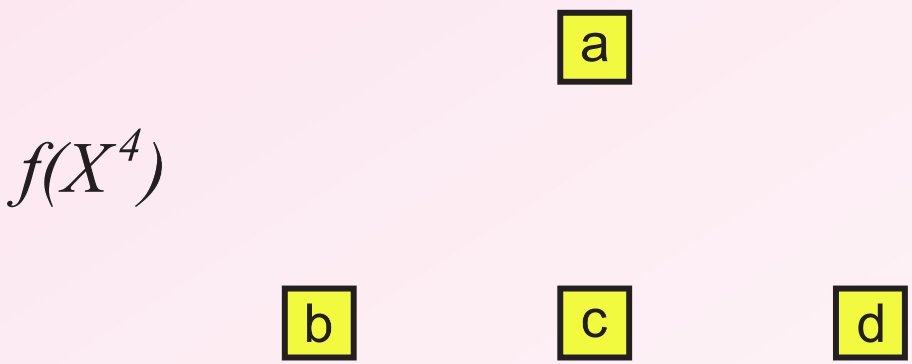

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Proof: a direct application of

$$f(X)^p \equiv f(X^p) \pmod{p\mathbb{Z}[X]}$$

for prime factors p of n.

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Lemma: $f(X^n) \in \mathrm{Ann}(c)$ for every $n \in \mathbb{N}$ whose prime factors are sufficiently large.

In particular, $f(X^{1+iM})$ are in $\mathrm{Ann}(c)$ for $i = 0, 1, 2, \dots,$ where M is the product of all small primes.

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Let $Z \in \mathbb{C}^d$ be a common zero of $\mathrm{Ann}(c)$, so

$$f(Z^{1+iM}) = 0 \text{ for all } i = 0, 1, 2, \ldots$$

Then (proof omitted) $g(Z) = 0$ for

$$g(X) = X^1 \prod_{\substack{I,J \in \mathcal{I} \\ I \neq J}} (X^{MI} - X^{MJ}).$$

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Nullstellensatz $\Longrightarrow g(X)^n \in \mathrm{Ann}(c)$ for some $n \in \mathbb{N}$.

Dividing $g(X)^n$ by a suitable monomial gives:

Theorem. Finitary, integral $c(X)$ that has a non-trivial annihilator is annihilated by a Laurent polynomial of the form

$$(1 - X^{I_1})(1 - X^{I_2}) \ldots (1 - X^{I_k}).$$

Annihilator: $(1 - X^{I_1})(1 - X^{I_2}) \ldots (1 - X^{I_k})$

Binomials $(1 - X^I)$ correspond to **difference operators** that subtract from a configuration its own I-translation.

The theorem states that configuration $c(X)$ can be annihilated by a sequence of difference operations.

Annihilator: $(1 - X^{I_1})(1 - X^{I_2}) \dots (1 - X^{I_k})$

If $k = 1$ then $c(X)$ is periodic.

More generally, $c(X)$ is a sum of k (possibly non-finitary) integral configurations that are periodic:

Corollary $c(X) = c_1(X) + \cdots + c_k(X)$ where $c_i(X)$ is I_i-periodic and **integral** (but not necessarily finitary).

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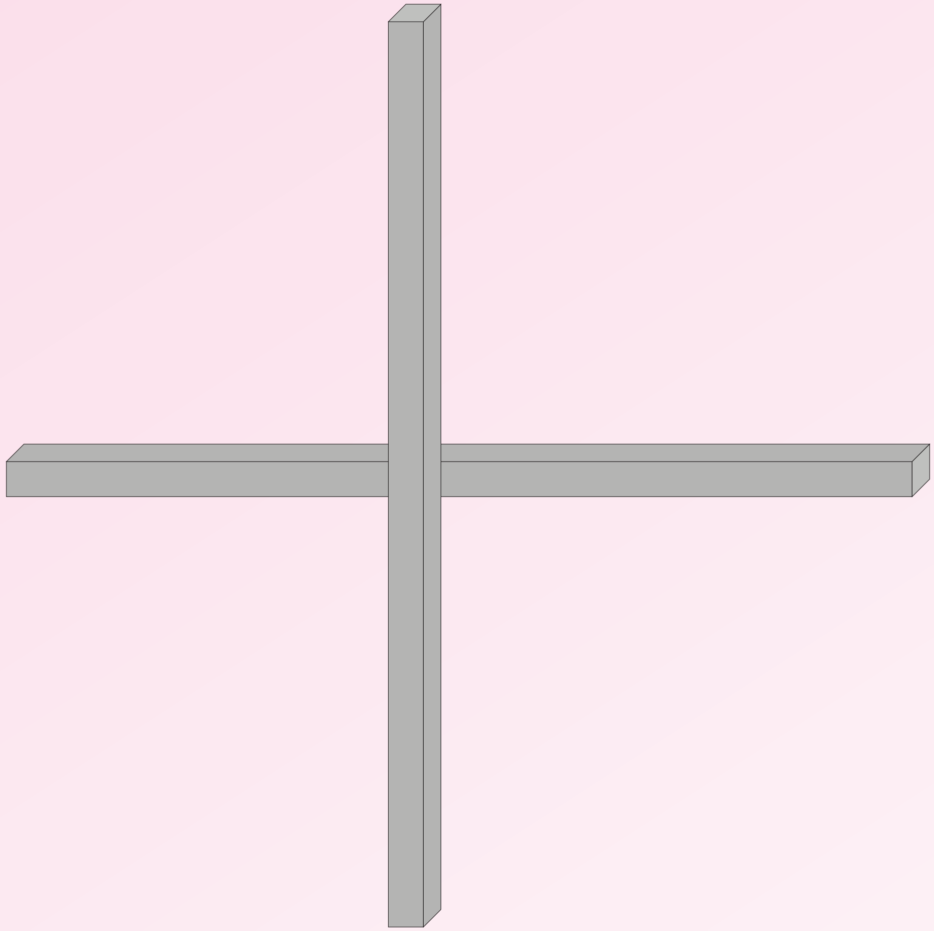
We can trade integrality for boundedness:

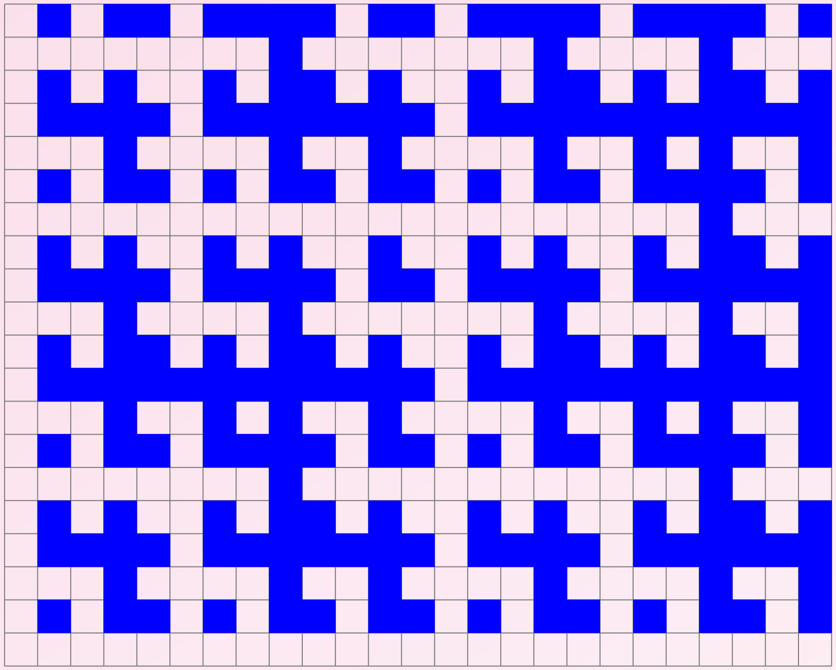
Corollary' $c(X) = c_1(X) + \cdots + c_k(X)$ where $c_i(X)$ is I_i-periodic with **bounded** coefficients (but not necessarily finitary).

Example. The 3D counter example



to Nivat's conjecture is a sum of two periodic configurations. It is annihilated by polynomial $(1-y)(1-x)$.

Another example. Sum of three non-finitary integral periodic configurations:



$$c(i,j) = \lfloor (i+j)\varphi \rfloor - \lfloor i\varphi \rfloor - \lfloor j\varphi \rfloor$$

Annihilated by $(x-1)(y-1)(xy^{-1}-1)$.

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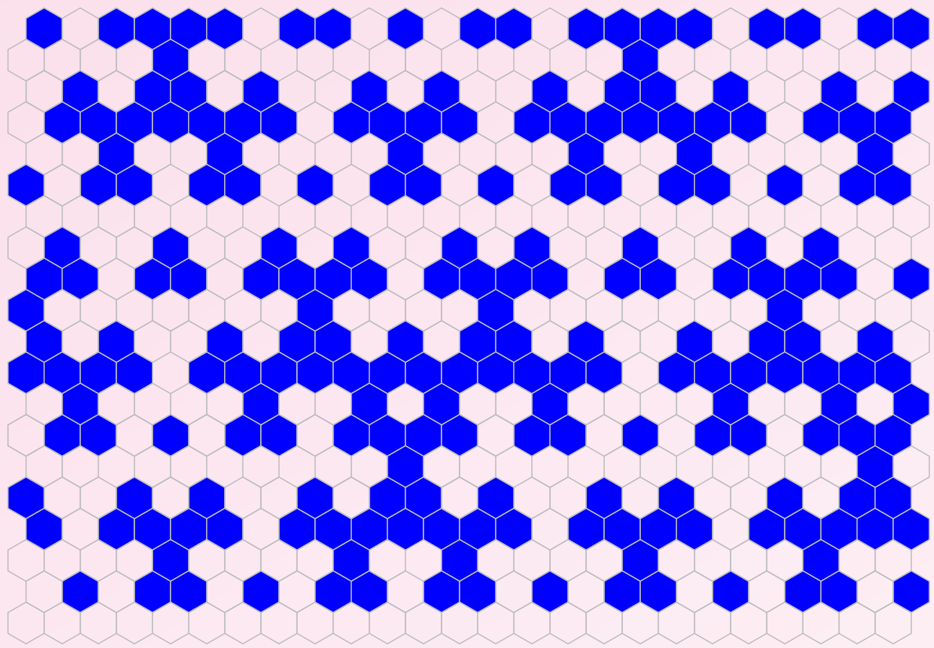
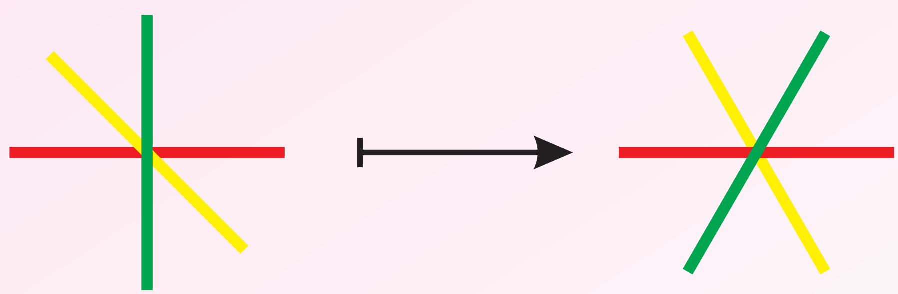


Nicer picture in skewed coordinates



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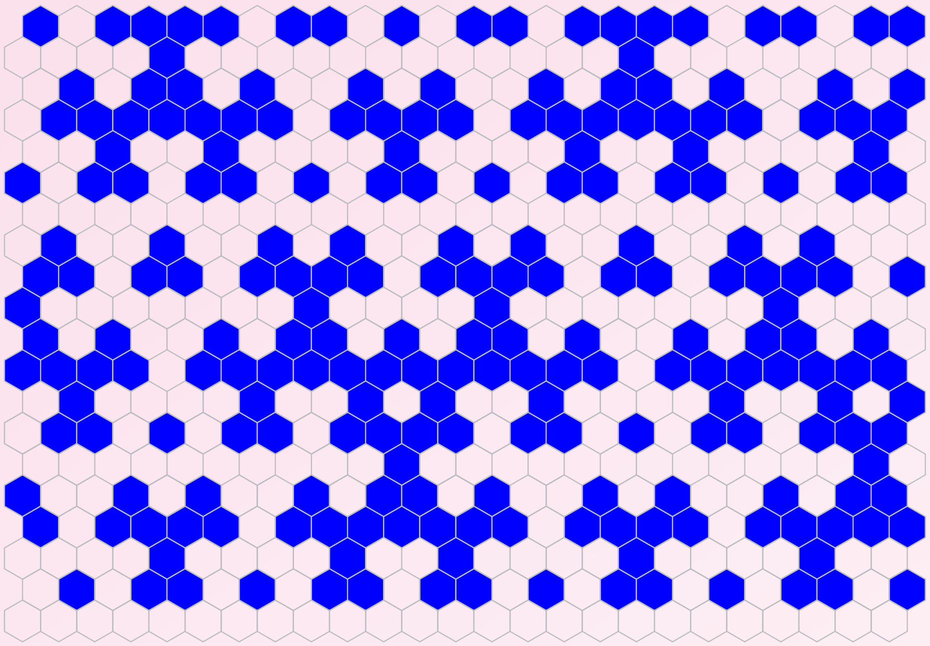
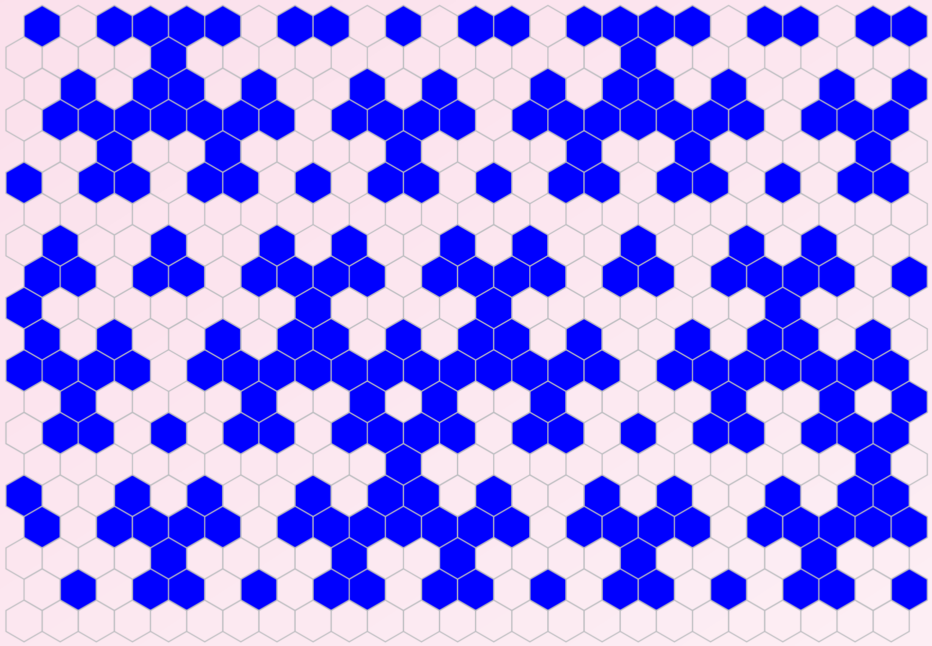


The configuration is also a sum of three periodic configurations with bounded (but non-integral) coefficients

$$\begin{aligned} c(i,j) &= (\lfloor (i+j)\varphi \rfloor - (i+j)\varphi) - (\lfloor i\varphi \rfloor - i\varphi) - (\lfloor j\varphi \rfloor - j\varphi) \\ &= -\{(i+j)\varphi\} + \{i\varphi\} + \{j\varphi\} \end{aligned}$$

Another example. Sum of three non-finitary integral periodic configurations:



Moreover: The configuration is not a sum of any number of periodic configurations that are finitary.

Our approach to Nivat's conjecture (simplified view)

Suppose $P(c, D) \leq |D|$ for some rectangle D.

Then c is annihilated by some

$$(1 - X^{I_1}) \ldots (1 - X^{I_k}).$$

Take the one with smallest k so directions I_i are pairwise different.

If $k = 1$ then c is periodic and we are done.

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$$(1 - X^{I_1}) \ldots (1 - X^{I_k}).$$

Take the one with smallest k so directions I_i are pairwise different.

Assume then $k \geq 2$. We want to prove that $P(c, X \times Y) > XY$ for all large rectangles $X \times Y$.

Filtering c with all but one of the k difference operators $1 - X^{I_i}$ provides us with $c_1, \ldots, c_k$ that are periodic only in directions $I_1, \ldots, I_k$, respectively.

First c_1:

Non-periodic sequence of stripes.

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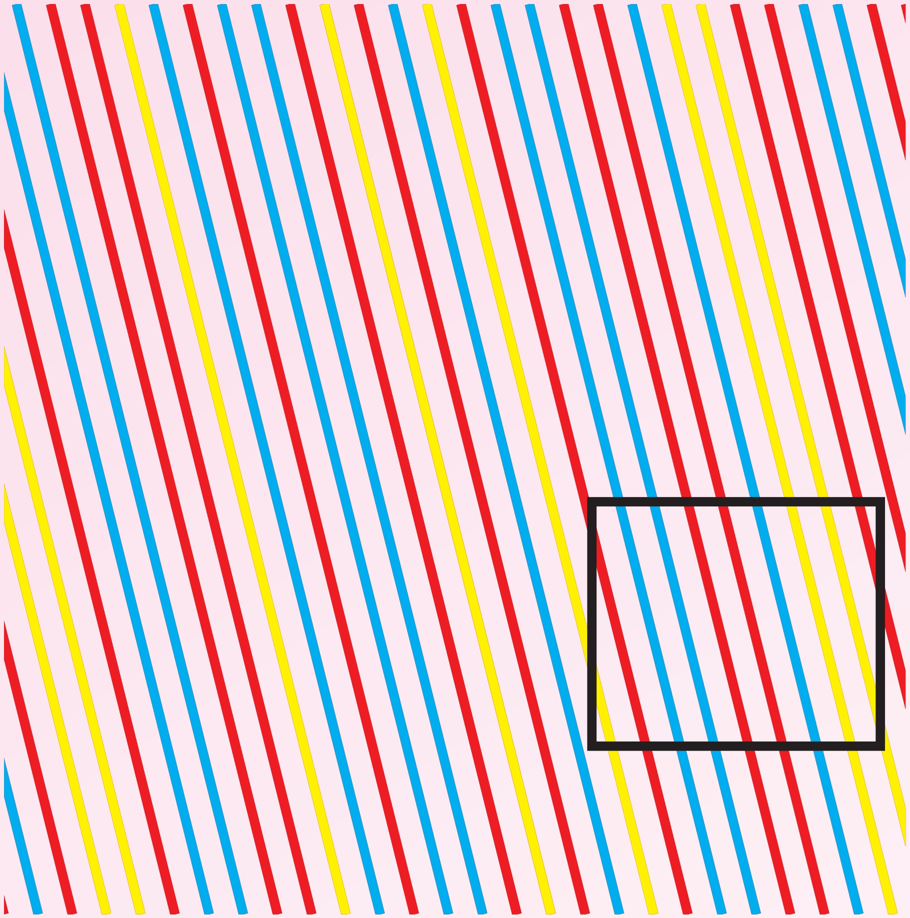


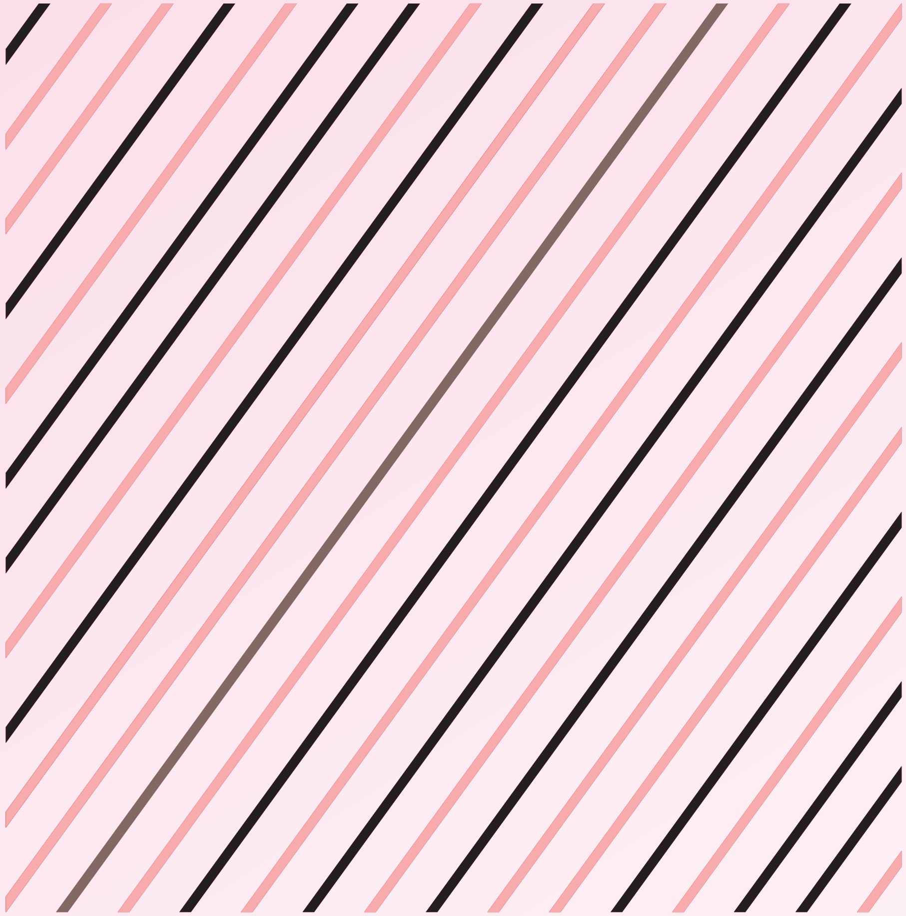
W.l.g. the stripes are not horizontal

$\Longrightarrow$ at least X stripes are visible in every $X \times Y$ rectangle

$\Longrightarrow$ more than X different $X \times Y$ blocks in c_1

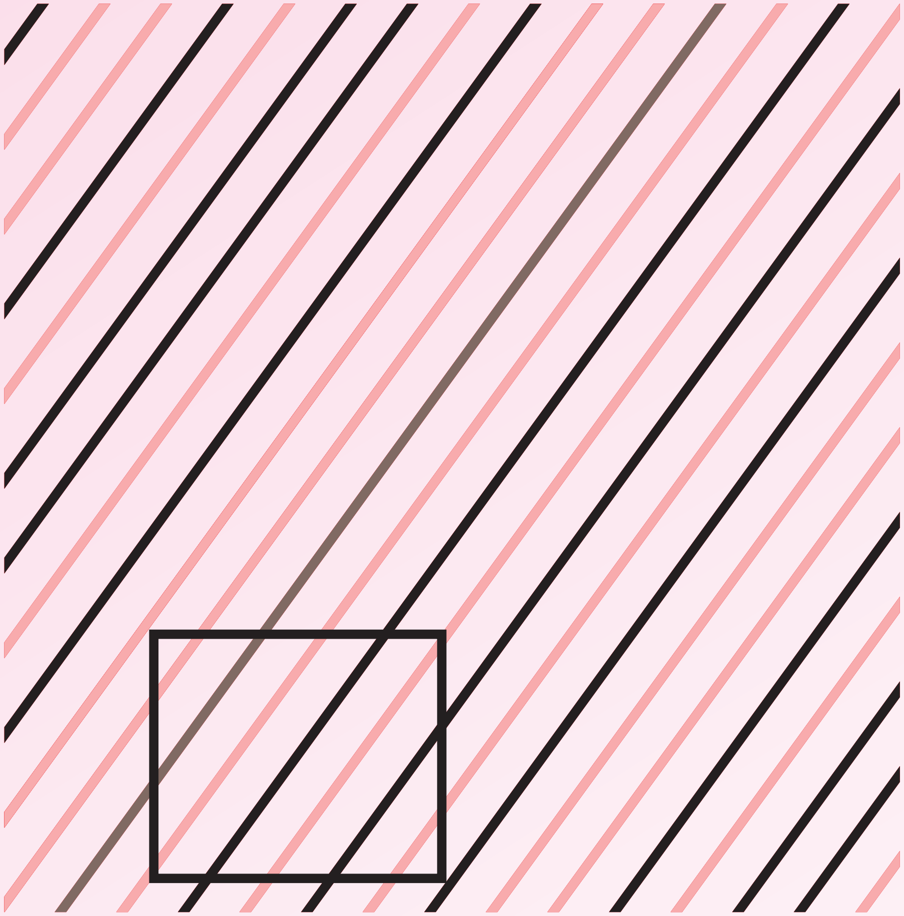
(due to the one-dimensional Morse-Hedlund theorem)

Second c_2:



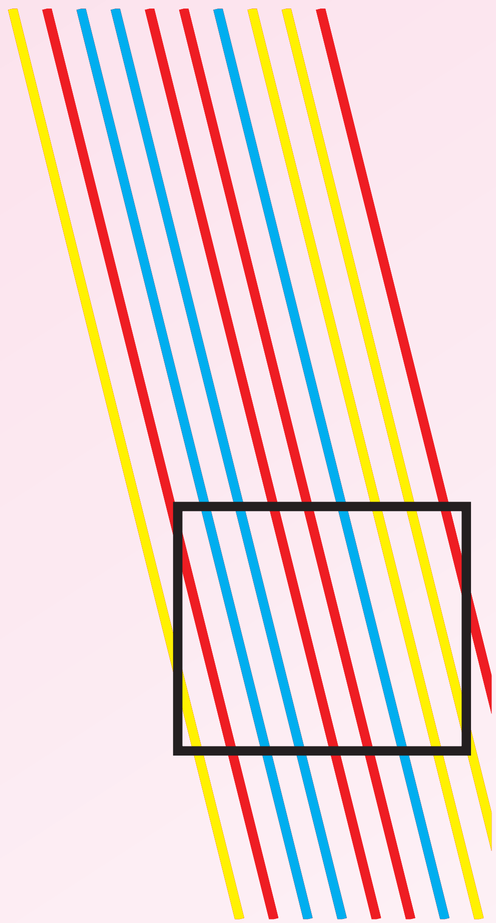
Non-periodic sequence of stripes in a different direction.

Second c_2:

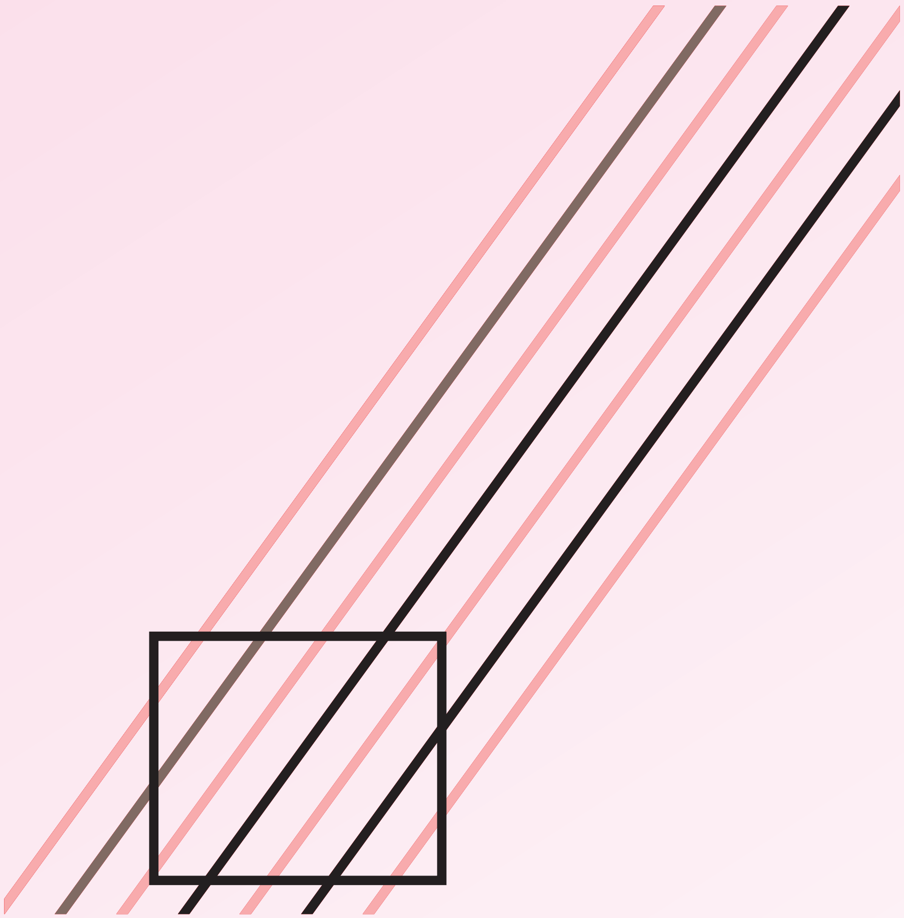


Analogously: stripes not vertical $\Longrightarrow$ more than Y different $X \times Y$ blocks in c_2).

Pick any $X \times Y$ pattern from $c_1 \ldots$



$\ldots$ and any $X \times Y$ pattern from c_2.



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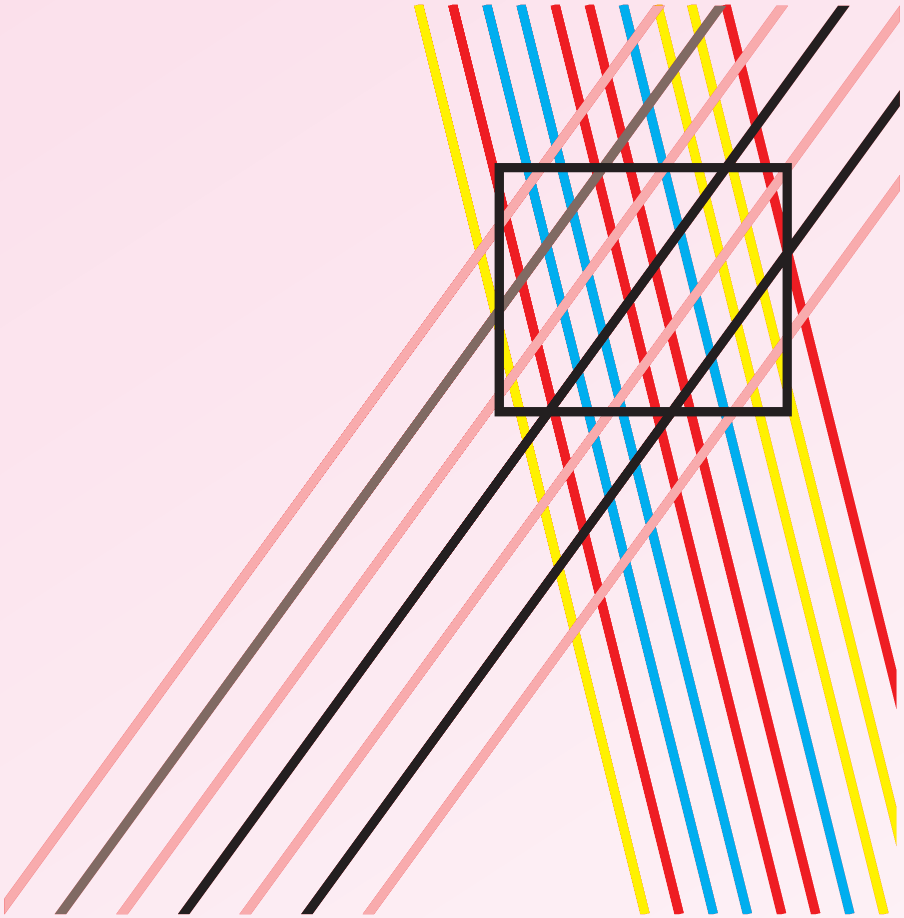

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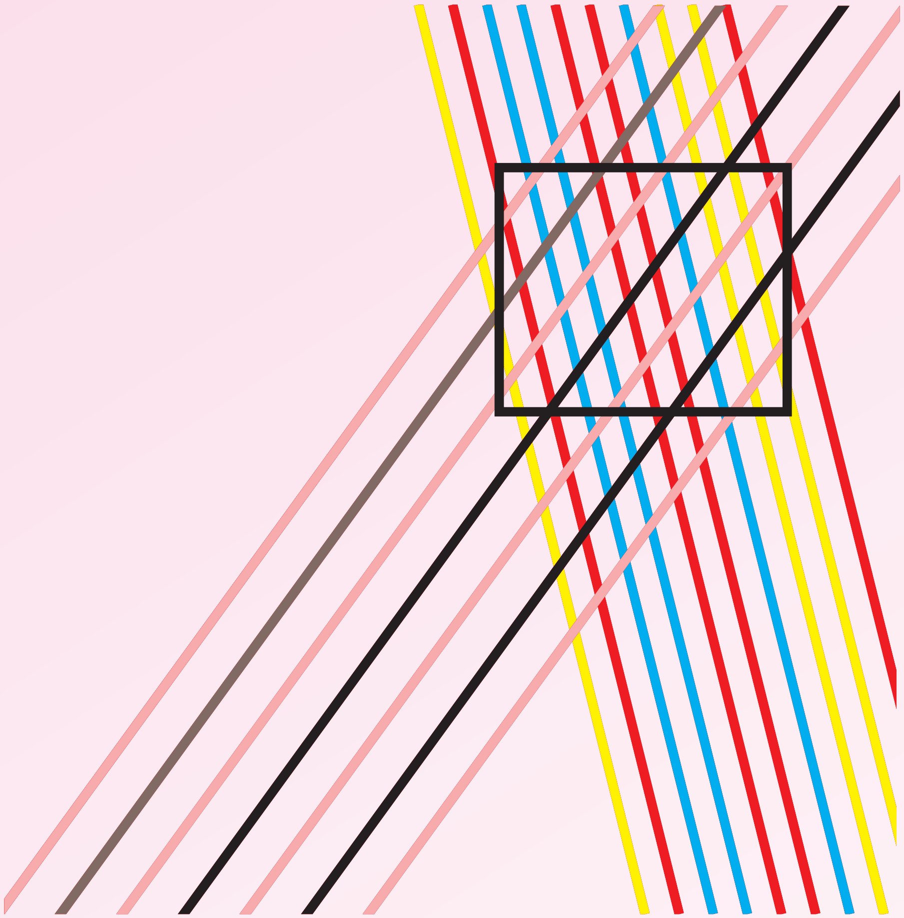

so both patterns can be seen (more or less) in the same position.

$\Longrightarrow$ more than XY distinct pairs of patterns in same positions

It follows (very roughly, skipping hugely many details)

Theorem. If c is a non-periodic 2D configuration then $P(c, D) \leq |D|$ can hold only for finitely many rectangles D.

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Questions:

- What can one say for other shapes than rectangles ? Perhaps an analogous result for convex shapes ?
- Can one use the periodic decomposition to address the periodic tiling problem ? What about other low complexity subshifts of finite type ?
- The original Nivat's problem is still open...

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Thank You